Math 115A
Linear Algebra

Midterm

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly and justify your answers. Please write your solutions in the space below the questions; you can go over the page and continue on the back; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name and UID in the space below.

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Number of additional sheets attached:

Question	Points	Score
1	10	10
2	10	9
3	10	10
4	10	8
Total:	40	37

Here are the axioms for vector spaces, in case you need them.

(VS 1) For all x, y in V, $x + y = y + x$.

(VS 2) For all x, y, z in V, $(x + y) + z = x + (y + z)$.

(VS 3) There exists an element in V denoted by $\underline{0}$ such that $x + \underline{0} = x$ for each x in V.

(VS 4) For each element x in V there exists an element y in V such that $x + y = \underline{0}$.

(VS 5) For each element x in V, $1x = x$.

(VS 6) For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$.

(VS 7) For each element a in F and each pair of elements x, y in V, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b in F and each element x in V, $(a + b)x = ax + bx$.

Problem 1.

Let V and W be vector spaces over a field F.

(a) [5pts.] Define what it means for a map $T\colon V \to W$ to be *linear*.

(b) [5pts.] Let $g(x) = e^x + e^{-x} \in \mathcal{F}(\mathbb{R}, \mathbb{R})$. Prove that the map

$$\begin{aligned} T\colon P_5(\mathbb{R}) &\to \mathcal{F}(\mathbb{R}, \mathbb{R}) \\ p(x) &\mapsto p(g(x)) \end{aligned}$$

is linear.

a) For a map $T: V \to W$ to linear, it must satisfy that

a) $\forall x, y \in V$

$T(x+y) = T(x) + T(y)$

b) $\forall x \in V$, $c \in F$

$T(cx) = c \cdot T(x)$.

b) a) ~~Let $f(x)$, $h(x) \in P_5(\mathbb{R})$, $c \in F$~~

~~$p(c \cdot f(x) + h(x))$~~

~~$= p$~~

~~Let $x, y \in P_5(\mathbb{R})$, $c \in F$~~

~~$T(p(cx+y)) = p(g(cx+y$~~

~~$T(cx+y) = g$~~

Let $f(x)$, $h(x) \in P_5(\mathbb{R})$

a) $\forall f(x), h(x)$

$T(f(x) + h(x))$

$= T((f+h)(x))$

$= (f+h)(g(x))$

$= f(g(x)) + h(g(x))$

$= T(f(x)) + T(h(x))$ #

Let $c \in F$, $f(x) \in P_5(\mathbb{R})$.

b) $\forall c \in F$, $f(x) \in P_5(\mathbb{R})$.

~~$T(c.f(x))$~~ $T((cf)(x))$

~~$= c \cdot f(g(x))$~~ $= (cf)(g(x))$

$= c \cdot f(g(x))$

$= c \cdot T(f(x))$ #

$\therefore$ by a) & b), T is a linear transformation from $P_5(\mathbb{R}) \to \mathcal{F}(\mathbb{R}, \mathbb{R})$

Problem 2.

Consider the linear map

$$T\colon P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$$
$$ax^2 + bx + c \mapsto \begin{pmatrix} c-a & b \\ b & a-c \end{pmatrix}$$

(a) [3pts.] Find the representation matrix of T with respect to the standard ordered bases $\beta = \{1, x, x^2\}$ and $\gamma = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$.

(b) [4pts.] Find $N(T)$ and its dimension.

(c) [3pts.] Find a basis of $R(T)$.

Remember to justify your answers. You do not need to prove that T is linear.

a) $[T]_\beta^\gamma = \begin{bmatrix} | & | & | & \cancel{|} \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & [T(v_3)]_\gamma & \cancel{[T(v_4)]_\gamma} \\ | & | & | & \cancel{|} \end{bmatrix}$

$[T(v_1)]_\gamma = [T(1)]_\gamma = \left[\begin{pmatrix} 1-0 & 0 \\ 0 & 0-1 \end{pmatrix}\right]_\gamma = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

$[T(v_2)]_\gamma = [T(x)]_\gamma = \left[\begin{pmatrix} 0-0 & 1 \\ 1 & 0-0 \end{pmatrix}\right]_\gamma = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$[T(v_3)]_\gamma = [T(x^2)]_\gamma = \left[\begin{pmatrix} 0-1 & 0 \\ 0 & 1-0 \end{pmatrix}\right]_\gamma = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]_\gamma = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

b). $N(T) = \{ x \in P_2(\mathbb{R}) \mid T(x) = \underline{0} \in M_{2\times 2}(\mathbb{R}) \}$.

$\underline{0} \in M_{2\times 2}(\mathbb{R}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{cases} c-a = 0 \\ b = 0 \\ b = 0 \\ a-c = 0 \end{cases} \Rightarrow \begin{cases} b = 0 \\ a = c \end{cases}$ is the criteria of $x \in P_2(\mathbb{R})$

Since there are only 1 variable that is free

$\Rightarrow x \in P_2(\mathbb{R})$: $ax^2 + \underline{0 \cdot x} + a$

$\dim(N(T)) = 1$.

c) Basis of $\boxed{R(T)}$

By theorem, given β a basis of $V(P_2(\mathbb{R}))$

$R(T) = \cancel{T(\beta)}\ \text{span}(T(\beta))$.

$$T(\beta) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\Rightarrow T(\beta)$ generates $R(T)$

~~for~~ To construct a Basis γ for $R(T)$, all we need to do is to pick all the linearly independent vectors in the generating set $T(\beta)$

$$\therefore \gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Show that this is the maximal linearly independent set!

$\therefore \gamma$ is the basis of $R(T)$.

2/3

Problem 3. *10pts.*

V

Let V and W be vector spaces over a field F, and let $T_0 \colon V \to W$ denote the zero map (i.e., $T_0(x) = \underline{0}$ for all $x \in V$). Let $T \colon V \to W$ be a linear map such that $T^2 = T_0$.

Prove that $T = T_0$ if and only if $R(T) \cap N(T) = \{\underline{0}\}$.

~~By Homework, we know that if $T^2 = T_0$~~

~~then $R(T) \subseteq N(T)$.~~

Part i) Claim: $R(T) \subseteq N(T)$ if $T^2 = T_0$ $T: V \to V$ linear.

$\forall$ $x \in V$, $T(x) \in R(T)$

$T(T(x)) = T^2(x) = T_0(x) = \underline{0}$.

Then by def of $N(T)$, $T(x) \in N(T)$.

$\Rightarrow$ $R(T) \subseteq N(T)$ # shown.

Proof.

Part ii) $\boxed{\Rightarrow}$ Suppose $T = T_0$

5

then $\forall x \in V$, $T(x) = T_0(x) = \underline{0}$

$\Rightarrow R(T) = \{\underline{0}\}$.

~~$\Rightarrow R(T) \cap N(T)$~~ $\because$ T is linear, $T(\underline{0}) = \underline{0}$

$\therefore$ $\underline{0} \in N(T)$ ✓ contain $\underline{0}$

Then, $R(T) \cap N(T) = \{\underline{0}\}$ # shown.

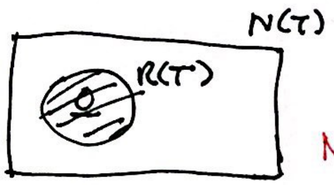
only has one element $\{\underline{0}\}$

$\boxed{\Leftarrow}$ Suppose $R(T) \cap N(T) = \{\underline{0}\}$.

5

then by part i), $R(T) \subseteq N(T)$ OK

$\Rightarrow R(T) = \{\underline{0}\}$ Yes



Nice picture

$\Rightarrow T = T_0$ if $R(T) = \{\underline{0}\}$

since $\forall x \in V$, $T(x) \in R(T) = \underline{0}$ # shown

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Problem 4.

(a) [5pts.] Let V be a vector space of finite dimension over a field F, and $\beta = \{v_1, \ldots, v_n\}$ be a basis of V. Suppose that $a_1, \ldots, a_n \in F$ are *non-zero* scalars. Prove that $\gamma = \{a_1v_1, \ldots, a_nv_n\}$ is a basis of V.

(b) [5pts.] Let V and W be finite-dimensional vector spaces over $\mathbb{C}$, and let $T: V \to W$ be an isomorphism. Prove that there are ordered bases β for V and γ for W such that all the entries of $[T]_\beta^\gamma$ are even integers.

4

a) First, $|\beta| = |\gamma| = n$

Then by corollary of replacement theorem, all I need to prove next is that γ generates V or γ is linearly independent. ✓

claim: γ generates V.

Since β is a basis of V

Let $x \in V$,

then $\exists\, b_1, \ldots, b_n \in F$ s.t. $x = b_1 v_1 + \ldots + b_n v_n$ not all $b_i = 0$ $\forall i$

maybe they are all 0 (e.g. for $x=0$) ✗

However, $\gamma = \{a_1 v_1, a_2 v_2, \ldots, a_n v_n\}$ where $a_1, \ldots, a_n \in F$ $\neq 0$

so, $x = \frac{b_1}{a_1}(a_1 v_1) + \ldots + \frac{b_n}{a_n}(a_n v_n)$ not all $\frac{b_i}{a_i} = 0$ $\forall i$ ✗ -1 pt

$\Rightarrow$ So for any arbitrary vector ~~it~~ x in V,

x can be written as a linear combination of vectors in γ set.

$\Rightarrow$ $\mathrm{span}(\gamma) = V$

$\therefore$ ~~span (γ = V)~~ $\mathrm{span}(\gamma) = V$ and $|V| = |\beta|$

$\therefore$ γ is a basis.

4

b) Since $T: V \to W$ is an isomorphism,
By theorem, $T(\beta)$ is a basis of W. OK

Small proof ~~incase~~ in case the theorem is not ~~allod~~ allowed:

Since T is isomorphism $\Rightarrow$ T is one-to-one & onto

$\Rightarrow$ one-to-one $\Rightarrow N(T) = \{0\}$

$\Rightarrow$ By rank-nullity theorem, $\dim(R(T)) = \dim(W) = \dim(V)$

basis of W has n elements

$|\beta| = |T(\beta)| = n.$

Furthermore, $T(\beta)$ is linearly independent. by the fact that β is ~~the~~ a basis and T is one to one

$\Rightarrow T(\beta)$ is a basis for W.

For any isomorphism linear $T: V \to W$. γ is a basis as $|\gamma| = |\beta|$

constructs the ordered γ basis as followingly γ is linearly independent. by a)

Let γ ~~$T(\beta)$~~ $= \{2v_1, 2v_2, \dots, 2v_n\}$ where $\beta = \{v_1, \dots, v_n\}$ is a basis of V.

Then, the claim is that $[T]_\beta^{\gamma}$ ~~$T(\beta)$~~ is a matrix such that all entries are even. $[T]_\gamma^{T(\beta)}$

Proof:

jth column of $[T]_\beta^{\gamma} = [T]_\beta^{\gamma} \cdot e_j = [T]_\beta^{\gamma} [v_j]_\beta$

$= [T(v_j)]_{\gamma}$ $=$ $[2v_j]_{\gamma}$ NO. T is not the multiplication by 2.

$= \begin{pmatrix} 0 \\ \vdots \\ 2 \\ \vdots \\ 0 \end{pmatrix}$ ← jth row.

, all other entries must be zero because $(2v_j)$ is linearly independent to all other vectors in γ.

$\Rightarrow$ ~~$[T]_\beta^{T(\beta)} =$~~

$$[T]_\beta^{\gamma} = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & & & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & \vdots & 0 & \cdots & 2 \end{bmatrix}$$

diagonal matrix of $2 \ddots 2$.

#shown.

Good idea, but the bases are confused.