Math 115A
Linear Algebra

Midterm

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly and justify your answers. Please write your solutions in the space below the questions; you can go over the page and continue on the back; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name and UID in the space below.

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Number of additional sheets attached: ____________

Question	Points	Score
1	10	10
2	10	10
3	10	10
4	10	9
Total:	40	39

Here are the axioms for vector spaces, in case you need them.

(VS 1) For all x, y in V, $x + y = y + x$.

(VS 2) For all x, y, z in V, $(x + y) + z = x + (y + z)$.

(VS 3) There exists an element in V denoted by $\underline{0}$ such that $x + \underline{0} = x$ for each x in V.

(VS 4) For each element x in V there exists an element y in V such that $x + y = \underline{0}$.

(VS 5) For each element x in V, $1x = x$.

(VS 6) For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$.

(VS 7) For each element a in F and each pair of elements x, y in V, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b in F and each element x in V, $(a + b)x = ax + bx$.

Problem 1.

Let V and W be vector spaces over a field F.

(a) [5pts.] Define what it means for a map $T\colon V \to W$ to be *linear*.

(b) [5pts.] Let $g(x) = e^x + e^{-x} \in \mathcal{F}(\mathbb{R}, \mathbb{R})$. Prove that the map

$$T\colon P_5(\mathbb{R}) \to \mathcal{F}(\mathbb{R}, \mathbb{R})$$
$$p(x) \mapsto p(g(x))$$

is linear.

a). $T: V \to W$ is linear. means that:

1) additive: $T(x+y) = T(x) + T(y)$ OK

2) homogeneous: $T(cx) = cT(x)$

b). 1). ~~To~~ $\forall x\, y \in P_5(R)$. $T(x+y) = T(x) + T(y)$.

1) $\forall$, $m, n \in P_5(R)$ $T(m+n) = T(m) + T(n)$

$m \in P_5(R)$ $\exists a_0 \cdots a_5$. $a_0 + \cdots + a_5x^5 = m$

$n \in P_5(R)$ $\exists b_0 \cdots b_5$ $b_0 + \cdots + b_5x^5 = n$.

$T(m+n) = T(a_0 + b_0 + \cdots + (a_5 + b_5)x^5)$.

$= a_0 + b_0 + \cdots + (a_5 + b_5)(e^x + e^{-x})^5$. $\cancel{= e^{a_0+b_0+\cdots+(a_5+b_5)x^5} + e^{-(a_0+b_0+\cdots+(a_5+b_5)x^5)}}$

$= a_0 + \cdots + a_5(e^x + e^{-x})^5$

$+ b_0 + \cdots + b_5(e^x + e^{-x})^5$.

$\cancel{= e^{a_0+\cdots+a_5x^5} + e^{-(a_0+\cdots+a_5x^5)} + e^{b_0+\cdots+b_5x^5} + e^{-b_0-\cdots-b_5x^5}}$

$\cancel{= T(m) + T(n)}$

$= T(m) + T(n)$ ✓

2). $\forall m \in P_5(R)$ $c \in F$ $T(cm) = cT(m)$

m (above) $= a_0 + \cdots + a_5x^5$ $cm = ca_0 + \cdots + ca_5x^5$

$\cancel{T(cm) = e^{c\cdot(a_0+\cdots+a_5x^5)} + e^{-c(a_0+\cdots+a_5x^5)}}$.

$\cancel{= c \cdot e^{a_0+\cdots+a_5x^5}}$

$\cancel{= cT(m)}$.

$T(cm) = c \cdot ca_0 + \cdots + ca_5(e^x + e^{-x})^5$

$= c \cdot (a_0 + \cdots + a_5(e^x + e^{-x})^5)$.

$= c \cdot T(m)$. ✓

5/5

or we don't have to write down what g is:

1). $T((p+q)(x))$

$= ((p+q) \circ g)(x)$

$= (p+q)(g(x))$.

$(e^x)^5 + e$

$= p(g(x)) + q(g(x))$

$= T(p(x)) + T(q(x))$.

2). $T(c \cdot p(x))$.

$= ((c \cdot p) \circ g)(x)$

$= (c \cdot p)(g(x))$

$= c \cdot (p(g(x)))$

$= c \cdot T(p(x))$

Problem 2.

Consider the linear map

$$T\colon P_2(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$$
$$ax^2+bx+c \mapsto \begin{pmatrix} c-a & b \\ b & a-c \end{pmatrix}$$

(a) [3pts.] Find the representation matrix of T with respect to the standard ordered bases $\beta = \{1, x, x^2\}$ and $\gamma = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$.

(b) [4pts.] Find $N(T)$ and its dimension.

(c) [3pts.] Find a basis of $R(T)$.

Remember to justify your answers. You do not need to prove that T is linear.

a). $[T]_\beta^\gamma = (T(1)_\gamma . \; T(x)_\gamma \; T(x^2)_\gamma .)$

$\vdash =$ $T(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad T(1)_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad T(x)_\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$T(x^2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad T(x^2)_\gamma = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$

b). $b=0 \quad a=c$

$N(T)$: $T(ax^2+bx+c) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

$c-a = a-c = 0 \rightarrow a=c \quad b=0$

$\therefore N(T) = \operatorname{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}.$ OK

dimension. $N(T) = 1.$

c). $\dim R(T) = 2.$

$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

$ax^2+bx+c \to \begin{pmatrix} c-a & b \\ b & a-c \end{pmatrix}$

$= \begin{pmatrix} c-a & 0 \\ 0 & a-c \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = c-a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

Basis of $R(T)$. : $\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$

Problem 3. *10pts.*

Let V and W be vector spaces over a field F, and let $T_0 \colon V \to \cancel{W}$ V denote the zero map (i.e., $T_0(x) = \underline{0}$ for all $x \in V$). Let $T \colon V \to W$ be a linear map such that $T^2 = T_0$.

Prove that $T = T_0$ if and only if $R(T) \cap N(T) = \{\underline{0}\}$.

① If $R(T) \cap N(T)$ $\{\underline{0}\}$, then. $T = T_0$.

$T^2 = T_0 \Leftrightarrow R(T) \subseteq N(T)$. (homework $\boxed{2.4}$ section).

proof: ~~$\forall x, T$~~ $\forall x \in V$: $T(T(x)) \in N(T)$ $m = T(x) \in R(t)$, $T(m) \in N(t)$. $R_t \subseteq N(t)$.

$R_t \subseteq N_t$ and $R_t \cap N_t = \{\underline{0}\}$: $R_t = \{\underline{0}\}$. $T = T_0$.

② If $T = T_0$. then $R(T) \cap N(T) = \{\underline{0}\}$:

1, $\{\underline{0}\} \in R_t$ $\{\underline{0}\} \in N(t)$ since both are basis.. $\underline{0} \in NT \cap RT$.

2, T : If $T = T_{(0)}$ $\forall x \in V$ $T(x) = \underline{0}$ $T(x) \in RT$. $\underline{0}$ and only $\underline{0} \in RT$.

$RT \cap NT = \{\underline{0}\}$

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Problem 4.

(a) [5pts.] Let V be a vector space of finite dimension over a field F, and $\beta = \{v_1, \dots, v_n\}$ be a basis of V. Suppose that $a_1, \dots, a_n \in F$ are *non-zero* scalars. Prove that $\gamma = \{a_1 v_1, \dots, a_n v_n\}$ is a basis of V.

(b) [5pts.] Let V and W be finite-dimensional vector spaces over $\mathbb{C}$, and let $T: V \to W$ be an isomorphism. Prove that there are ordered bases β for V and γ for W such that all the entries of $[T]_\beta^\gamma$ are even integers.

a) prove γ:

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① $\gamma \subseteq V$. $|\gamma| = |\beta| = \dim V = n$.

Try to prove that: ~~γ is linearly~~ span $\gamma = V$.

$\forall x \in V$ $\exists m_1 \cdots m_n$ $m_1 v_1 + \cdots + m_n v_n = x$.

$\forall m_1 \cdots m_n$, $\exists b_1 \cdots b_n$, $b_1 a_1 = m_1 \cdots b_n a_n = m_n$. since $a_1 \cdots a_n$ are non zero.

$\forall x \in V$ $\exists b_1 \cdots b_n$ $b_1(a_1 v_1) + \cdots + b_n(a_n v_n) = x$.

~~span~~ γ $V \subseteq$ span γ span $\gamma = V$.

a basis of V.

b) choose $\beta = \{v_1 \cdots v_n\}$ to be an arbitrary basis of V.

~~choose $\gamma = \{\frac{1}{2} T$~~

$\forall T: V \to V$ isomorphism

choose $\gamma = \{\frac{1}{2} T(v_1) \cdots \frac{1}{2} T(v_n)\}$

) T isomorphism $\to$ γ is a basis

) $[T]_\beta^\gamma = \begin{pmatrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{pmatrix}$

all entries are even integers.

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b). $T: V \to W$ is an isomorphism.

Define: $T: V \to W$ be:

$T: \mathbb{C}^2 \to \mathbb{C}^2$ } isomophism
$v_1 \to 2v_2$

$[T]_\beta^\gamma = \cancel{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} \begin{pmatrix} 2&0&0&0 \\ 0&2&0&0 \\ 0&0&2&0 \\ 0&0&0&2 \end{pmatrix}$

Not a basis of $\mathbb{C}^2$ over $\mathbb{C}$

(This is a basis of $\mathbb{C}^2$ over $\mathbb{R}$)

$\beta = \{\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}i\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}0\\i\end{pmatrix}\}$.

$\gamma = \{\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}i\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}0\\i\end{pmatrix}\}$

开始思路走偏了，举例子，而非general proof

① Better remember the examples from hw.

② After fixing β, γ, we I should compute $[T]_\beta^\gamma$ just from ~~β and~~ $T(v_i)_\gamma$, not, generalizing it to $T(x)_\gamma$.

$[T]_\beta^\gamma \cdot [v]_\beta = [T(v)]_\gamma$.

there exists a bases β and γ, such that.)

$T(v_1)_\gamma \cdots T(v_n)_\gamma$. are all even.

($\beta = \{v_1 \cdots v_n\}$).

$T: V \to W$ is an isomorphism:

$\dim V = n$, then $|\beta| = |\gamma| = 2n$. NO

(for each dimension, we need. $\begin{pmatrix}0\\ \vdots\\1\\ \vdots\\0\end{pmatrix} \begin{pmatrix}0\\ \vdots\\i\\ \vdots\\0\end{pmatrix}$).

~~define~~ $\beta = \{v_1 \cdots v_n\}$ $\to$ ~~standard basis.~~ can be any basis of V

$\gamma = \{\frac{1}{2} T(v_1) \cdots \frac{1}{2} T(v_n)\}$. $\leftarrow$ Good choice
$\subseteq W$.

If T is an isomorphism, then β is a basis of V, then γ is a basis for W. Yes

Let $x = a_1 v_1 + \cdots + a_n v_n$.

$T(x) = a_1 T(v_1) + \cdots + a_n T(v_n)$.

$T(x)_\gamma = \begin{pmatrix} 2a_1 \\ \vdots \\ 2a_n \end{pmatrix}$ Yes

~~$T(x)_\gamma$ is always even.~~ NO $T(x)_\gamma$ is not always even, but $T(v_i)_\gamma$ is always even.

$2(T(v_i)) = 2(T(v_i)_\gamma)$. $T(v_i)_\gamma$ is always even.

$T(v_i)_\gamma$ is even.: $T(v_1)_\gamma \cdots T(v_n)_\gamma$ are even $[T]_\beta^\gamma$ is even.

$[T(v_i)_\gamma \ \forall i]$ is even. ~~$T(v)$~~ $[T]_\beta^\gamma$ is even.

Yes but why? (Key observation: $[T(v_i)]_\gamma = \begin{pmatrix}0\\2\\0\end{pmatrix}$).

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