Math 115A
Linear Algebra

Midterm

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly and justify your answers. Please write your solutions in the space below the questions; you can go over the page and continue on the back; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name and UID in the space below.

First name: Ziqing

Last name: Jiang

Student ID number: 204917652

Number of additional sheets attached: ____________

Question	Points	Score
1	10	10
2	10	4
3	10	10
4	10	9
Total:	40	33

Here are the axioms for vector spaces, in case you need them.

(VS 1) For all x, y in V, $x + y = y + x$.

(VS 2) For all x, y, z in V, $(x + y) + z = x + (y + z)$.

(VS 3) There exists an element in V denoted by $\underline{0}$ such that $x + \underline{0} = x$ for each x in V.

(VS 4) For each element x in V there exists an element y in V such that $x + y = \underline{0}$.

(VS 5) For each element x in V, $1x = x$.

(VS 6) For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$.

(VS 7) For each element a in F and each pair of elements x, y in V, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b in F and each element x in V, $(a + b)x = ax + bx$.

Problem 1.

Let $P =$ "$\forall n \in \mathbb{N}, \exists m \in \mathbb{Z}, n \cdot (n+1) = 2m$".

(a) [3pts.] Write the negation $\neg P$ (your answer must not contain the negation symbol $\neg$).

(b) [7pts.] Prove P by induction.

a) $\exists n \in N \ \forall m \in Z \ \ n \cdot (n+1) \neq 2m$

b) Base case: $n=1$. $1 \cdot (1+1) = 2 = 2 \cdot 1$. $m = 1 \in Z$. Base case is true

Inductive step: Suppose: $k \in n \ \exists m \in Z, \ k(k+1) = 2m$.

prove : $\exists m' \in Z$ $\cancel{(k+1)(k+2) = 2m'}$. $(k+1)[(k+1)+1] = 2m'$

$(k+1)(k+2) = k^2 + 3k + 2 = (k(k+1) + 2(k+1)) = 2m + 2(k+1) = 2(m+k+1)$

$m' = m+k+1 \in Z$.

Inductive step is true.

Thus: $\forall n \in N, \exists m \in Z, n(n+1) = 2m$.

Problem 2.

Recall that $\mathcal{F}(\mathbb{R},\mathbb{R})$ is the vector space over $\mathbb{R}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) [2pts.] What is the zero vector $\underline{0}$ of $\mathcal{F}(\mathbb{R},\mathbb{R})$? Why?

(b) [8pts.] Let $g \in \mathcal{F}(\mathbb{R},\mathbb{R})$ be a fixed function from $\mathbb{R}$ to $\mathbb{R}$, and define

$$W = \{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f \circ g = \underline{0}\}.$$

Prove that W is a subspace.

~~$g(f(x)) = \underline{0}$~~

$f(g(x)) = \underline{0}$

a) $F(x) = 0$. $F(R, 0)$.

$\forall S \exists$:

$\forall$ ~~F~~ $f \in F(R,R)$.

~~$x + 0 = x$~~. $f + \underline{0} = xf$.

1/2

Explanation is unclear

$F(X) = x$. $\forall x \in R$. , $y \in R$ $f(x) = y$ $F(x,0) = 0$

$F(X + 0) = x + 0$ $f(x) + F(x,0) = y + 0 = y$.

$F(X) + F(x) = X + 0$ ~~F~~ $F(x, 0)$ is the zero vector.

$F(x) = 0$.

1/2 b). ① $\underline{0} \in W$. $\underline{0}$ in V is. $F(x) = \underline{0}$ let. ~~$x = g(x)$~~ $\forall X = g(x)$. ~~$F(X)$~~ $F(g(x)) = F(X) = \underline{0}$.

1/3 ② $\forall x, y \in W$ $x + y \in W$ Then ~~let~~ $X(g(x)) = \underline{0}$ $Y(g(x)) = \underline{0}$ $X(g(x)$ $X + Y = X + Y(g(x)) = \underline{0}$. $\boxed{X + Y \in W}$ $((X+Y)\circ g)(x)$

1/3 ③ $\forall a \in F$ $x \in W$ $aX \in W$. Then ~~let~~ $X(g(x)) = \underline{0}$ $a \cdot X(g(x)) = a \cdot \underline{0} = \underline{0}$ $a \cdot X \in W$.

① $\underline{0} \in W$ from a) we know that $f(x) = 0$ is $\underline{0}$ in $F(R,R)$.
~~$\forall g(x) \in R$ $f \circ g = f(g(x))$~~ $\forall x \in R$ let $m = g(x)$ $f(m) = 0$ thus $f(x) = 0 \in W$.
② $\forall x, y \in W$, $X + y \in W$. $X(g(x)) = 0$ $Y(g(x)) = 0$ $(X+Y)(g(x)) = X(g(x)) + Y(g(x)) = 0 + 0 = 0$ $X+Y \in W$
③ $\forall a \in F$ $x \in W$. $aX \in W$. $X(g(x)) = 0$ $(aX)(g(x)) = a \cdot X(g(x)) = a \cdot 0 = 0$ $aX \in W$

Scanned with CamScanner

Problem 3. *10pts.*

Let V be a vector space over a field F, and let $v \in V$. Prove that $\{v\}$ is linearly dependent if and only if $v = \underline{0}$.

~~Suppose that $\{v\}$ is linearly dependent~~:

~~$\exists a \in F$, $a \neq 0$ $av = \underline{0}$~~

If $v = \underline{0}$, $\{v\}$ is linearly dependent:

5

Let $a = 1$, $a \in F$: $a \cdot \vec{v} = 1 \cdot \underline{0} = \underline{0}$ (VS5)

$a \neq 0$, this is a non-trivial linear combination

$\{v\}$ is linearly dependent.

5

If $\{v\}$ is linearly-dependent, $v = \underline{0}$

Prove by contradiction :

Suppose that $v \neq 0$. ~~and $\{v\}$ is linearly dependent.~~

$\forall a \in F, a \neq 0$ $a \cdot v \neq \underline{0}$., $\{v\}$ is linearly independent.

Thus, v must be $\underline{0}$., in order to have $a \neq 0$, $av = \underline{0}$.

let $V = \begin{Bmatrix} V_1 \\ \vdots \\ V_n \end{Bmatrix}$. $V_1 - V_n$ not all 0.

$a \cdot V = \begin{Bmatrix} aV_1 \\ \vdots \\ aV_n \end{Bmatrix}$ $aV.$ aV_n not all 0

$a \cdot V \neq \underline{0}$

Only works in F^n

But the fact that $v \neq 0$ and $a \neq 0$ $\rightarrow$ $av \neq 0$ is true in general

Problem 4.

Recall that the sum of two subspaces W and Z of V is defined as

$$W + Z = \{w + z \mid w \in W, z \in Z\}.$$

Let V be a vector space over F, and let $S_1, S_2 \subseteq V$ be subsets.

(a) [3pts.] Give an example where $\text{span}(S_1 \cup S_2) \neq \text{span}(S_1) \cup \text{span}(S_2)$.

(b) [7pts.] Prove that $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

3

(a) $S_1 = \{\binom{0}{1}\}$ $S_2 = \{\binom{1}{0}\}$ V=?

$S_1 \cup S_2 = \{\binom{0}{1}, \binom{1}{0}\}$.

$\text{Span}(S_1) : \{ \binom{0}{a} \mid \forall a \in F\}$.

$\text{span}(S_2) : \{ \binom{a}{0} \mid \forall a \in F\}$.

$\text{span}(S_1 \cup S_2) \{ \binom{a}{b} \mid \forall a, b \in F\}$.

Eg.: $\binom{1}{1} = 1 \cdot \binom{0}{1} + 1 \binom{1}{0} \in \text{Span}(S_1 \cup S_2)$

$\binom{1}{1} \notin \text{Span}(S_1)$ $\binom{1}{1} \notin \text{Span}(S_1) \cup \text{Span}(S_2)$

$\binom{1}{1} \notin \text{Span}(S_2)$

b) ~~$\text{Span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) + \text{Span}(S_2)$.~~

~~Let $S_1 = \{v_1 \cdots v_n\}$. $S_2 = \{v_n \cdots v_{n+m}\}$ (can overlap)~~

~~$\forall v \in \text{Span}(S_1 \cup S_2)$. $S_1 \cup S_2 = \{ \cdots \}$.~~

~~$\exists a_1 \cdots a$.~~

6 × −1

① $\text{span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) + \text{Span}(S_2)$.

$S_1 = \{v_{1,1} \cdots v_{1,n}\}$ $S_2 = \{v_{2,1} \cdots v_{2,n}\}$. not necessarily finite

$S_1 \cup S_2 = \{u_1 \cdots u_m\}$.

$\forall v \in \text{Span}(S_1 \cup S_2)$ $\exists a_1 \cdots a_m$ $a_1 u_1 + \cdots + a_m u_m = v$.

$\forall u_i \in (S_1 \cup S_2)$ $u_i \in S_1$ or $u_i \in S_2$.

Thus, we can divide $S_1 \cup S_2$ into 2 groups where $u_1 \cdots u_k \in S_1$, $u_{k+1} \cdots u_m \in S_2$.

Now chose arbitrary $k < m$ & $u_1 \sim u_k \in S_1$ $u_{k+1} \sim u_m \in S_2$.

Then $a_1 u_1 + \cdots a_k u_k \in \text{Span } S_1$

$a_{k+1} u_{k+1} + \cdots + a_m u_m \in \text{Span} S_2$.

$\text{Span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) + \text{Span}(S_2)$

② $\text{Span}(S_2) + \text{Span}(S_1) \subseteq \text{Span}(S_1 \cup S_2)$.

$\forall v \in \text{Span } S_1$. $\exists a_1 \cdots a_n$.

$a_1 v_{1,1} + \cdots + a_n v_{1,n} = v$

$\forall w \in \text{Span}. S_2$. $\exists b_1 \cdots b_n$

$b_1 v_{2,1} \cdots + b_n v_{2,n} = w$.

$v + w = a_1 v_{1,1} + \cdots + a_n v_{1,n}$ What is v?

$v_{i,j}$? $+ b_1 v_{2,1} + \cdots + b_n v_{2,n}$

$\forall v_{i,j}$ $v \in S_1$ or $v \in S_2$, $v \in S_1 \cup S_2$?

$v + w \in \text{Span}(S_1 \cup S_2)$.

Thus. $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$

Scanned with CamScanner