Math 115A
Linear Algebra

Midterm

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly and justify your answers. Please write your solutions in the space below the questions; you can go over the page and continue on the back; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name and UID in the space below.

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Number of additional sheets attached: __________

Question	Points	Score
1	10	10
2	10	10
3	10	10
4	10	9
Total:	40	39

Here are the axioms for vector spaces, in case you need them.

(VS 1) For all x, y in V, $x + y = y + x$.

(VS 2) For all x, y, z in V, $(x + y) + z = x + (y + z)$.

(VS 3) There exists an element in V denoted by $\underline{0}$ such that $x + \underline{0} = x$ for each x in V.

(VS 4) For each element x in V there exists an element y in V such that $x + y = \underline{0}$.

(VS 5) For each element x in V, $1x = x$.

(VS 6) For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$.

(VS 7) For each element a in F and each pair of elements x, y in V, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b in F and each element x in V, $(a + b)x = ax + bx$.

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$k \cdot (k+1)$

k^2+k

Problem 1.

Let $P =$ "$\forall n \in \mathbb{N}, \exists m \in \mathbb{Z}, n \cdot (n+1) = 2m$".

(a) [3pts.] Write the negation $\neg P$ (your answer must not contain the negation symbol $\neg$).

(b) [7pts.] Prove P by induction.

a) $\neg P =$ "$\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, n \cdot (n+1) \neq 2m$"

b) let $P(x)$ be P when $n = x$

when $n = 1$, $1 \cdot (1+1) = 2$

$\Rightarrow$ value for $m = 1$

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. $k(k+1) = 2m'$, $m' \in \mathbb{Z}$

when $n = k+1$, L.H.S $= (k+1)(k+1+1)$

$$= (k+1)(k+2)$$

$$= k^2 + 3k + 2$$

$$= (k^2+k) + 2k+2$$

$$= k(k+1) + 2(k+1)$$

$$= 2m' + 2(k+1)$$

$$= 2(m'+k+1)$$

$\therefore$ $P(k+1)$ is true when $P(k)$ is true.

$\therefore$ by induction, P is true for all $n \in \mathbb{N}$.

Problem 2.

Recall that $\mathcal{F}(\mathbb{R},\mathbb{R})$ is the vector space over $\mathbb{R}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) [2pts.] What is the zero vector $\underline{0}$ of $\mathcal{F}(\mathbb{R},\mathbb{R})$? Why?

(b) [8pts.] Let $g \in \mathcal{F}(\mathbb{R},\mathbb{R})$ be a fixed function from $\mathbb{R}$ to $\mathbb{R}$, and define

$$W = \{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f \circ g = \underline{0}\}.$$

Prove that W is a subspace.

a). It is the zero function, i.e $f(x) = 0$.
2/2
This is because the zero vector $= (0 \cdot f)$, so any output from the function must be zero.
Also, $g + f = g$ for any $g \in F(\mathbb{R},\mathbb{R})$.

b) ① Prove $\underline{0} \in W$
2/2
$\because$ when $f = \underline{0}$, $f \circ g = \underline{0}$ ✓
$\therefore$ $\underline{0} \in W$

② Prove $\forall a, b \in W$, $a+b \in W$

3/3
$\because$ $a \circ g = \underline{0}$
$b \circ g = \underline{0}$
$\therefore$ $(a+b) \circ g = a \circ g + b \circ g = \underline{0} + \underline{0} = \underline{0}$
$\therefore$ $a+b \in W$.

③ Prove $\forall c \in F$, $a \in W$, $ca \in W$
3/3
$\because$ $a \circ g = \underline{0}$
$\therefore$ $(ca) \circ g = c(a \circ g) = c\,\underline{0} = \underline{0}$

$\therefore$ $ca \in W$

$\therefore$ W is a subspace

Problem 3. *10pts.*
Let V be a vector space over a field F, and let $v \in V$. Prove that $\{v\}$ is linearly dependent if and only if $v = \underline{0}$.

$\boxed{\Rightarrow}$ (B.C) Suppose that $\{v\}$ is linearly dependent, but $v \neq \underline{0}$

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$\because$ $\{v\}$ is linearly dependent

$\exists\, a \in F$, $a \neq 0$, $av = \underline{0}$

$\because$ $v \neq \underline{0}$, the only possible solution is $a = 0$

$\rightarrow$ contradiction. Thus v must be $\underline{0}$.

$\boxed{\Leftarrow}$ $\because$ $v = \underline{0}$

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$\therefore$ One solution for $av = \underline{0}$, $a \in F$, $a \neq 0$.

is $a = 1$.

$\therefore$ $\{v\}$ is linearly dependent

Problem 4.

Recall that the sum of two subspaces W and Z of V is defined as

$$W + Z = \{w + z \mid w \in W, z \in Z\}.$$

Let V be a vector space over F, and let $S_1, S_2 \subseteq V$ be subsets.

(a) [3pts.] Give an example where $\text{span}(S_1 \cup S_2) \neq \text{span}(S_1) \cup \text{span}(S_2)$.

(b) [7pts.] Prove that $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

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a). $S_1 = \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$, $S_2 = \left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$ in $V = F^2$

$\text{Span}(S_1 \cup S_2) = \text{Span}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}a\\b\end{pmatrix} \,\middle|\, a, b \in F\right\}$

$\text{Span}(S_1) \cup \text{Span}(S_2) = \left\{\begin{pmatrix}a\\0\end{pmatrix} \,\middle|\, a \in F\right\} \cup \left\{\begin{pmatrix}0\\b\end{pmatrix} \,\middle|\, b \in F\right\} \neq \text{Span}(S_1 \cup S_2)$

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b) Let $S_1 = \{u_1, u_2, \ldots, u_n\}$

$S_2 = \{v_1, v_2, \ldots, v_{n'}\}$, $S_1 \cup S_2 = \{u_1, \ldots, u_n, v_1, \ldots, v_{n'}\}$

Why finite? −1pt

$\text{Span}(S_1 \cup S_2) = \{a_1 u_1 + a_2 u_2 + \ldots + a_n u_n + b_1 v_1 + b_2 v_2 + \ldots + b_{n'} v_{n'}\}$, $a_i, b_j \in F$, $1 \leq i \leq n$, $1 \leq j \leq n'$

$\text{Span}(S_1) = \{a_1' u_1 + a_2' u_2 + \ldots + a_n' u_n\}$

$\text{Span}(S_2) = \{b_1' v_1 + b_2' v_2 + \ldots + b_{n'}' v_{n'}\}$

$a_i', b_j' \in F$, $1 \leq i \leq n$, $1 \leq j \leq n'$

$\text{Span}(S_1) + \text{Span}(S_2) = \{w + z \mid w \in \text{Span}(S_1), z \in \text{Span}(S_2)\}$

$= \{a_1' u_1 + a_2' u_2 + \ldots + a_n' u_n + b_1' v_1 + b_2' v_2 + \ldots + b_{n'}' v_{n'}\}$

$\therefore \text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$