BRUIN ACTUARIAL SOCIETY

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ge, a list of notation
ıake sure your exam
includes each page.

Please write your name on *each* page you submit.

You will have 50 minutes to complete this exam. You may not use a calculator, or consult your textbook, class notes, or any other materials. If you need more space for your answers, please use the back of the pages.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in chapters 1-5 of the textbook, or facts from Math 33A, but please make it clear when you are doing so.

Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Question	Points	Score
1	30	21
2	20	10
3	20	20
4	15	2
5	15	0
Total:	100	53

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Recall the following notation (here, F is a field):

- $\mathbb{R}$ is the field of real numbers.
- $\mathbb{C}$ is the field of complex numbers.
- $\mathbb{Q}$ is the field of rational numbers.
- $F^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in F\}$ is the vector space of n-dimensional vectors with entries in F.
- $M_{m \times n}(F)$ is the vector space of $m \times n$ matrices (i.e. m rows n columns) with entries in F.
- $P(F)$ is the vector space of all polynomials with coefficients in F.
- $P_n(F)$ is the vector space of all polynomials of degree $\leq n$.
- For any set S, $\mathcal{F}(S, F)$ is the vector space of functions from S to F.
- $C(\mathbb{R})$ is the vector space of *continuous* functions $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$.
- If V and W are vector spaces, $\mathcal{L}(V, W) = \{T : V \to W | T \text{ is linear}\}$ is the vector space of linear transformations $T : V \to W$.

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1. [30 pts] Let $V = P_2(\mathbb{R})$ and let $\beta = \{1+x, x+x^2, 1+x^2\}$ be an ordered basis for V. (You do NOT need to show that this is a basis.)

$2x^2 = b + c - a \quad 2 = a + c - b$

$2x = a + b - c$

12 **(a)** [12 pts] Let $T : V \to V$ be the linear transformation defined by $T(f(x)) = f(0) + xf'(x)$. Find $[T]_\beta$ (you do NOT need to show that T is linear)

$T(1+x) = 1 + x =$ $a = 1$, $b = 0$, $c = 0$

$T(x+x^2) = 0 + (1+2x)x = x + 2x^2 = \frac{1}{2}(a+b-c) + b + c - a$

$a = -\frac{1}{2}$ $b = \frac{3}{2}$ $c = \frac{1}{2}$

$T(1+x^2) = 1 + (2x)x = 1 + 2x^2 = \frac{1}{2}(a+c-b) + b + c - a$

$a = -\frac{1}{2}$ $b = \frac{1}{2}$ $c = \frac{3}{2}$

$$[T]_\beta = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

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6 **(b)** [6 pts] Let W be a two dimensional vector space over $\mathbb{R}$, and let $\gamma = \{w_1, w_2\}$ be an ordered basis for W. Let $S : V \to W$ be a linear transformation, and assume that $[S]_\beta^\gamma = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$. Find $[ST]_\beta^\gamma$, where T is the linear transformation from part (a) .

$$[S]_\beta^\gamma [T]_\beta^\beta$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{pmatrix}$$

$$\checkmark = \boxed{\begin{pmatrix} 1 & -1 & -2 \\ 2 & 1/2 & -1/2 \end{pmatrix}}$$

3 **(c)** [12 pts] Let $\alpha = \{1, x, x^2\}$ be another ordered basis for V. Find the matrix Q which changes α-coordinates into β-coordinates (that is, the matrix Q for which $Q[f(x)]_\alpha = [f(x)]_\beta$ for all $f(x) \in V$).

$\beta = \{1+x, x+x^2, 1+x^2\}$

$$[I]_\beta^\alpha$$

$1 + x$ ~~4~~ $=$ $a = 1$ $b = 1$ ~~c~~ $= 0$

$x + x^2 =$ $a = 0$ $b = 1$ $c = 1$

$1 + x^2 =$ $a = 1$ $c = 1$

$$\boxed{Q = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$ this is ~~the~~ Q^{-1} !

2. [20 pts] Let $A = \begin{pmatrix} 3 & 3 & -9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

5 **(a)** [5 pts] Find all of the eigenvalues of A, along with their algebraic multiplicities.

$$\begin{vmatrix} 3-\lambda & 3 & -9 \\ 0 & 1-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)(3-\lambda) \checkmark$$

$\boxed{\lambda = 1,\ 3 \text{ mult } 2}$

4 **(b)** [10 pts] For each of the eigenvalues you found in part (a), find the dimension of the corresponding eigenspace.

$\lambda = 1$, $\left[\begin{array}{ccc|c} 2 & 3 & -9 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

dimension $= 1$ why? of what?

$\lambda = 3$, $\begin{bmatrix} 0 & 3 & -9 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

dimension $= 1$ ✓ 2

) **(c)** [5 pts] Is A diagonalizable? Explain why or why not. (If it is diagonalizable, you do NOT need to give an eigenbasis or find a matrix Q such that QAQ^{-1} is diagonal, simply showing it is diagonalizable is sufficient.)

for $\lambda = 1$, $n - \text{rank}(A - I\lambda)$ $\begin{bmatrix} 2 & 3 & -9 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

$3 - 2 = 1$ ✓ mult $= 1$ ✓

no

for $\lambda = 3$, $n - \text{rank}(A - \lambda I)$ $\begin{bmatrix} 0 & 3 & -9 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$3 - 2 = 1$

since should be yes multiplicity is one but is two. A isn't diagonalizable.

20/20

3. [20 pts] Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by

$$T(f(x)) = \begin{pmatrix} f(0) & f'(1) \\ f'(-1) & f(2) \end{pmatrix}$$

(you do NOT need to prove that T is linear).

8/8

(a) [8 pts] Is T onto? Prove that your answer is correct.

No

rank T = dim W to be onto.

by rank nullity thm: dim V = rank T + nullity T

from which, we can extract dim V ≥ rank T.

since dim W = 4 > 3 = dim V ≥ rank T, we know that dim W > rank T.

Thus, not onto.

12/12

(b) [12 pts] Is T one-to-one? Prove that your answer is correct.

Yes

$f(x) = a + bx + cx^2$

$$T(f(x)) = \begin{pmatrix} a & b+2c \\ b-2c & a+2b+4c \end{pmatrix}$$

nullity(T) = 0, then one to one

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 \end{array}\right]$$

② + ③
② - ③

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 \end{array}\right]$$

since nullity(T) = 0, T is one to one.

2/15

SI
T is
SU

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x 7 0
1x = 4(0)=C

4. [15 pts] Let V and W be finite dimensional vector spaces and assume that $\dim V > \dim W$. Let $T : V \to W$ and $U : W \to V$ be linear transformations. Prove that the composition $UT : V \to V$ is *not* an isomorphism.

UT is an isomorphism iff for basis a over V and B over V, $[UT]_a^B$ has an inverse by thm.

However, by definition of change in coordinate matrix, we know $[AB]_a^B = [A]_\lambda^B [B]_a^\lambda$ and $[U]_\lambda^B [T]_a^\lambda$ where λ is a basis in W.

Nope. This formula only works for square matrices.

so $([UT]_a^B)^{-1} = ([U]_\lambda^B [T]_a^\lambda)^{-1}$ since $(AB)^{-1} = B^{-1}A^{-1}$

$= ([T]_a^\lambda)^{-1} ([U]_\lambda^B)^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$ only works if you already know A & B have inverses.

~~but we know only square matrices have inverses and that $[T]_a^\lambda$ and $[U]_\lambda^B$ are not square since $\dim(a) = \dim(B) > \dim(\lambda)$ by definition of change in coordinate matrix. since basis is not of same dim for conversion. Thus, since $[UT]_a^B$ has no inverse, then UT has no inverse and $UT : V \to V$ is not an isomorphism.~~

0/15

Nar[redacted]

) so
) is
eigenvector
w/ eigenvalue
λ

5. [15 pts] Let V be a vector space and let $T : V \to V$ and $U : V \to V$ be linear. Assume that $TU = UT$ (that is, that T and U *commute*). Let v be an eigenvector of T, and assume that $U(v) \neq 0$. Prove that $U(v)$ is also an eigenvector of T.

if v is an eigen vector of T, then

Tv

$Tv = v\lambda$

~~$T(v)\lambda = aT(v)$~~

$U^{-1}T = TU^{-1}$

$T(v)\,U(v) = U(v)\,T(v)$ assuming $U(v) \neq 0$, $\exists$ a $U^{-1}(v)$

$U^{-1}(v)\,Tv = v\lambda U^{-1}$

$U^{-1}(v)\,T_v\,U(v) = v\lambda$

$U^{-1}(v)\,U(v)\,T\cancel{v} = \lambda\cancel{v}$

$U^{-1}(v)\,T\,U(v) = \lambda$

This statement is nonsense. $U^{-1}(v)TU(v)$ isn't a scalar.

since λ is an eigenvalue,

$U^{-1}(v)$ is a vector.

U

I never said U was invertible

$U(v)$ = eigenvector

Are you multiplying two vectors?

$Tv = \lambda v$

$T(U(v)) = U(T(v))$

$= U(\lambda v) = \lambda U(v)$

$U(v) \neq 0$ so $U(v)$ is eigenvector of T w/ eval λ

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