1. Let V be a vector space.

(a) (7 points) True or false: If S is a linearly dependent subset of V then each vector in S is a linear combination of the other vectors in S?
You must justify your answer to receive credit!

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False. For example, $\{1, x, x+x^2, x^2, x^3\} \in V$.

Since x, x^2 and $x+x^2$ are l.d.

The set is also l.d.

but x^3 cannot be expressed as a linear combination of the other vectors.

(b) (8 points) If S and S' are two subsets of V. Prove that

0.

$$span(S \cup S') = span(span(S) \cup span(S'))$$

$\dim(S) = \dim(span(S))$

$\dim(S \cup S') = \dim(span(S) \cup span\ S')$

$\Rightarrow span(S \cup S') = span(span(S) \cup span(S'))$

is not a subspace

2. (a) (7 points) Let V be a vector space and $\{u, v, w\}$ a basis of V. Prove that $\{u+v, u+w, v+w\}$ is also a basis of V.

$\because \{u, v, w\}$ is a basis of V.

$\dim(V) = 3$

for $\{u+v, u+w, v+w\}$ suppose $a, b, c \in F$

$a(u+v) + b(u+w) + c(v+w) = 0$.

$(a+b)u + (a+c)v + (b+c)w = 0$.

$\because u, v, w$ l.i.

$\begin{cases} a+b=0 \\ a+c=0 \\ b+c=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0. \end{cases}$

Since $\{u+v, u+w, v+w\}$ is l.i and $n=3$.

It's a basis of V.

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(b) (8 points) Find a basis for $P := \{(x, y, z) \in \mathbb{R}^3;\ 2x + 3y + 5z = 0\}$.

$x = 1$ $\quad$ $x = 0$

$y = -\frac{2}{3}$ $\quad$ $y = 1$

$z = 0.$ $\quad$ $z = -\frac{3}{5}$

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$\{(1, -\frac{2}{3}, 0), (0, 1, -\frac{3}{5})\}$ is a basis for P

3. (a) (10 points) Let V be a vector space, $S := \{v_1, v_2, \ldots, v_n\}$ is a spanning set of V and $S' = \{u_1, u_2, \ldots, u_m\}$ is another spanning set of V. Suppose that $n > m$. Show that the S is not a basis of V.

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Suppose S is l.i.

Since S is a spanning set.

we have non-zero sets of $a_1, a_2, \ldots, a_n \in \mathbb{F}$ that

$u_1 = a_1 v_1 + a_2 v_2 + \cdots a_n v_n$.

$u_2 = a_1' v_1 + a_2' v_2 + \cdots + a_n' v_n$.

$\vdots$

$u_m = a_1^{(m)} v_1 + a_2^{(m)} v_2 + \cdots + a_n^{(m)} v_n$.

$\because m < n$ and $S' = \{u_1, \cdots u_m\}$ is also a spanning set.

for at least one vector v_1

we can express it in. $v_1 = b_1 u_1 + b_2 u_2 + \cdots + b_m u_m$

with some $b_i \neq 0$. $1 \leq i \leq m$

since u_1, u_2, u_m can be expressed in terms of v_1 ...

we have a contradiction that S is not l.i.

Thus not a basis.

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$= v_1$, in

hat case

am (*)

don't

4. Let A be the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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and let T be the linear transformation defined by $Tx := Ax$.

(a) (8 points) Find bases for both $N(T)$ and $R(T)$.

$T(x) = 0$. (a, b, c)

$a + c$
$a + b + c$
b.

$a + c = 0$,
$a + b + c = 0$,
$b = 0$,
$a = -c$.

$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ basis of $N(T)$.

$\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}$. basis of $R(T)$.

(b) (2 points) Compute the nullity and rank of T.

$\dim(N(T)) = 1$.

$\operatorname{rank}(T) = \dim(V) - \dim(N(T))$
$= 3 - 1$
$= 2$.