Problem 1.

Decide whether each of the following sets V with the operations of addition and scalar multiplication specified is a vector space. Justify your answers.

(a) [5pts.] $V \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is the set of 2×2 matrices with determinant zero, and the operations inherited from $\mathrm{Mat}_{2\times 2}(\mathbb{R})$. (Recall that if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix, the determinant is $ad - bc$.)

Solution: Notice that this set is not preserved by addition:

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0, \text{ but } \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

Ergo it is not a subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, and in particular not a vector space.

(b) [5pts.] $V = \{(a,a_2) : a_1, a_2 \in \mathbb{R}\}$ with operations

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 - 3b_2)$$
$$c(a_1, a_2) = (ca_1, c^2 a_2)$$

Solution: V is not a vector space because (e.g.) addition fails to commute. Notice that $(0,0) + (1,1) = (2,-3)$, but $(1,1) + (0,0) = (1,1)$.

Problem 2.

Consider the subset V of polynomials $P_2(\mathbb{R})$ such that, for any $ax^2 + bx + c$ in V, we have $a + b + c = 0$.

(a) [5pts.] Prove that V is a subspace of $P_2(\mathbb{R})$.

Solution:

- The additive identity, $0x^2 + 0x + 0$, is in V.
- If $ax^2 + bx + c$ and $ex^2 + fx + g$ are in V, then $a + b + c = 0 = e + f + g$. Hence their sum $(a+e)x^2 + (b+f)x + (g+c)$ has $(a+e) + (b+f) + (g+c) = (a + b + c) + (d + e + f) = 0$, and is in V.
- If $ax^2 + bx + c \in V$, then $a + b + c = 0$. So if $h \in \mathbb{R}$, the scalar product $h(ax^2 + bx + c) = (ha)x^2 + (hb)x + (hc)$ has $ha + hb + hc = h(a + b + c) = 0$, and is in V.

(b) [5pts.] Find the dimension of V.

Solution: We see that any ax^2+bx+c in V can be rewritten $ax^2+bx+(-a-b) = a(x^2-1)+b(x-1)$. Therefore $\beta = \{x^2-1, x-1\}$ spans V, and is also clearly linearly independent since its two elements have different degrees, so β is a basis for V. Hence the dimension of V is two.

Problem 3.

Consider the set $S = \{(2,3,5), (1,0,-1), (-2,1,7), (1,4,11)\} \subset \mathbb{R}^3$.

(a) [5pts.] Is S linearly independent or linearly dependent? Justify your answer *without doing a computation.*

Solution: The dimension of $\mathbb{R}^3$ is three, so any linearly independent set in $\mathbb{R}^3$ must have no more than three elements. Ergo S must be linearly dependent.

(b) [5pts.] Find a subset of S that is a basis for $\mathbb{R}^3$.

Solution: We build a maximal linearly independent subset for $\mathbb{R}^3$. First, $\{(2,3,5)\}$ is linearly independent because it is a set consisting of a single nonzero vector. Next, $\{(2,3,5),(1,0,-1)\}$ is linearly independent because neither vector is a multiple of the other. Finally, consider $\beta = \{(2,3,5),(1,0,-1),(-2,1,7)\}$. If some linear combination $a(2,3,5)+b(1,0,-1)+c(-2,1,7) = 0$, we have

$$\begin{cases} 2a+b-2c=0 \\ 3a+c=0 \\ 5a-b+7c=0 \end{cases}$$

In any nontrivial solution, we must have $c = 0$ (because the set $\{(2,3,5),(1,0,-1)\}$ is linearly independent) so after possibly scaling we can assume $c=1$. Hence our equations become

$$\begin{cases} 2a+b=2 \\ 3a=-1 \\ 5a-b=-7 \end{cases}$$

The second equation gives $a = -13$, so by the first equation $b = \frac{8}{3}$.But then the last equation becomes $-\frac{13}{3} = -7$. So no nontrivial solution exists. Hence β is linearly independent and, having three elements, is a basis for $\mathbb{R}^3$.

Problem 4.

Let S_1 and S_2 be subsets of a vector space V.

(a) [5pts.] Prove that $\text{span}(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$.

> **Solution:** Suppose that $v \in \text{span}(S_1 \cap S_2)$, so $v = a_1u_1 + \cdots + a_nu_n$ is a linear combination of vectors $u_1, \cdots, u_n \in S_1 \cap S_2$. Then since each u_i is an element of S_1, v is also a linear combination of elements of S_1, hence $v \in \text{span}(S_1)$. Similarly, $v \in \text{span}(S_2)$. So $v \in \text{span}(S_1) \cap \text{span}(S_2)$. Since v was arbitrary, $(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$.

(b) [5pts.] Give an example in which the sets above are equal and one in which they are unequal.

> **Solution:** For equality, consider $S_1 = \{(1,0),(0,1)\}$ and $S_2 = \{(1,0\}$ in $\mathbb{R}^2$. Then $\text{span}(S_1) = \mathbb{R}^2$, and $\text{span}(S_2)$ is the x-axis. Then $\text{span}(S_1) \cap \text{span}(S_2)$ is the x-axis as well, so since $S_1 \cap S_2 = \{(1,0)\}$, $(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$. For inequality, let $S_1 = \{(1,0),(0,1)\}$ in $\mathbb{R}^2$ and $S_2 = \{(1,1)\}$. We see that $\text{span}(S_1 \cap S_2) = \text{span}(\phi) = \{0\}$. But $\text{span}(S_1) \cap \text{span}(S_2) = \mathbb{R}^2 \cap \text{span}(S_2) = \text{span}(S_2)$ is the line $\text{span}(\{(1,1)\})$.

Problem 5.

Recall that if W_1 and W_2 are subspaces of a vector space V, then

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

If in addition $W_1 \cap W_2 = \emptyset$, then we call this space $W_1 \oplus W_2$. If $W_1 \oplus W_2 = V$, then W_2 is said to be the complement of W_1.

(a) [5pts.] Prove that the xy-plane and the z-axis are complements in $\mathbb{R}^3$.

> **Solution:** The xy-plane is the subspace $\{(x,y,0) : x,y \in \mathbb{R}$ and the z-axis is the subspace $\{(0,0,z) : z \in \mathbb{R}\}$. These subspaces certainly have intersection $\{(0,0,0)\}$, and moreover any $(x,y,z) \in \mathbb{R}^3$ may be expressed as $(x,y,0) + (0,0,z)$.

(b) [5pts.] Let V be an n-dimensional vector space, and W_1 a k-dimensional subspace of V. Prove that W_1 has a complement; that is, prove that there exists W_2 such that $W_1 \oplus W_2 = V$. [Hint: Start with a basis for W_1, and extend to a basis for V. Now you should be able to find a candidate basis for W_2.]

> **Solution:** Let $\{x_1, \cdots, x_k\}$ be a basis for W_1, and extend to a basis $\beta = \{x_1, \cdots, x_k, y_1, \cdots, y_{n-k}\}$ for V. Then let $W_2 = \text{span}(\{y_1, \cdots, y_{n-k}\})$. We claim that $V = W_1 + W_2$. For any $v \in V$ is a linear combination of elements of β, and therefore may be written $a_1x_1 + \cdots a_kx_k + b_1y_1 + \cdots b_{n-k}y_{n-k} = (a_1x_1 + \cdots a_kx_k) + (b_1y_1 + \cdots b_{n-k}y_{n-k})$, a sum of elements in $W_1 = \text{span}(\{x_1, \cdots, x_k\})$ and $W_2 = \text{span}(\{y_1, \cdots, y_{n-k}\})$. Ergo $V = W_1 + W_2$. Moreover, sup-

pose that $v \in W_1 \cap W_2$. Then we may write $v = a_1x_1 + \cdots + a_kx_k$, because $v \in W_1$, but also $v = b_1y_1 + \cdots + b_{n-k}y_{n-k}$, because $v \in W_2$. So $0 = a_1x_1 + \cdots + a_kx_k - b_1y_1 - \cdots - b_{n-k}y_{n-k}$. Since β is a basis, this implies all of the a_i and b_j are in fact zero, and we conclude that $v = 0$. So $W_1 \cap W_2 = \{0\}$, and we see $V = W_1 \oplus W_2$.