MATH 115A MIDTERM 1

Due at 11:59 PM PST, Friday 4/23

Instructions

Complete and submit solutions to the problems below. The solutions that you submit should be in pdf form (handwritten, electronically written, or LaTeX-generated is fine). If your solutions are handwritten, use an app like CamScanner to produce a clear and legible pdf.

When you write up solutions, you should focus on writing a **clear, easy to follow, well-justified** argument. To give you an extreme-sounding perspective: treat each solution as a technical essay whose purpose is to convince the reader of the claim. You should use proofs from the textbook, proofs and solutions that I write in class, and proofs and solutions that Ben writes in discussion as references for good mathematical writing.

The exam is open-note, open-book, open-whatever. At the end of the day, **the solutions you submit should be yours and yours alone.**

Problems (27 points total)

1. (7 points) Consider the vector space $\mathbb{R}^4$. Let

$$v_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},\ v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$

Let $W := \operatorname{span}(v_1, v_2, v_3)$. Compute, with proof, $\dim W$.[1]

2. Recall that $M_n(\mathbb{R})$ is the (real) vector space of $n \times n$ matrices with real entries. Recall that if $A \in M_n(\mathbb{R})$, then A_{ij} is the (i, j)th entry of A. Let

$$W_1 = \{ A \in M_n(\mathbb{R}) \mid A_{ij} = 0 \text{ if } i > j \}$$
$$W_2 = \{ A \in M_n(\mathbb{R}) \mid A_{ij} = 0 \text{ if } i < j \}$$

be the subspaces of upper-triangular matrices and lower-triangular matrices, respectively.[2]

(a) (4 points) Prove that $M_n(\mathbb{R}) = W_1 + W_2$.

(b) (4 points) Prove that $M_n(\mathbb{R})$ is *not* the direct sum of W_1 and W_2.

3. For each of the following statements, determine whether it is **true** or **false**. If it is true, prove it. If it is false, give an explicit counter example or an explicit reason why.

(a) (4 points) Let V be a vector space and let $W_1, W_2 \subset V$ be subspaces. If $\mathcal{B}_1$ is a basis for W_1 and $\mathcal{B}_2$ is a basis for W_2, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $W_1 + W_2$.

(b) (4 points) Let $F(\mathbb{R}, \mathbb{R})$ be the (real) vector space of functions $f : \mathbb{R} \to \mathbb{R}$. The set

$$W = \{ f \in F(\mathbb{R}, \mathbb{R}) \mid f(x) \leq 1 \text{ for all } x \in \mathbb{R} \}$$

is a subspace of $F(\mathbb{R}, \mathbb{R})$.

(c) (4 points) Let V be a vector space. If $S_1, S_2 \subset V$ are (nonempty) subsets of V such that $S_2 \subset \operatorname{span}(S_1)$, then $S_1 \cup S_2$ is linearly dependent.

[1] Remember that we have not defined or used row reduction at all in 115A!

[2] You don't have to prove that they are subspaces, you can assume this.

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Problems (27 points total)

1. (7 points) Consider the vector space $\mathbb{R}^4$. Let

$$v_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$

Let $W := \text{span}(v_1, v_2, v_3)$. Compute, with proof, $\dim W$.[1]

Solution. Since the definition of dimension is the length of a basis, we need to find a basis for W. We submit the following

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\} = \{v_2, v_3\}$$

For $\mathcal{B}$ to be a basis, it must fulfill two properties

- $\mathcal{B}$ is linearly independent.
  To prove this, we simply must show that if there are scalars a and b such that

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = 0$$

  Then $a = b = 0$. To show this, we apply basic vector multiplication

$$\begin{pmatrix} 1a \\ 1a \\ 0a \\ 2a \end{pmatrix} + \begin{pmatrix} 1b \\ -1b \\ 1b \\ 0b \end{pmatrix} = 0$$

  Then we apply multiplicative identity, vector addition, and the lemma that $0 * a = 0 \forall a$ to obtain

$$\begin{pmatrix} a+b \\ a-b \\ b \\ 2a \end{pmatrix} = 0$$

  We recall that for a vector to be equal to the zero element, all of its components must be equal to the 0 element. This means we can extract

$$b = 0$$

$$2a = 0$$

  From row 3 and 4 of the vector. We see that b must be 0.

$$2a = 0$$

[1]Remember that we have not defined or used row reduction at all in 115A!

By multiplicative inverse.

$$a = 0/2$$

By the lemma that 0 multiplied by any number is 0.

$$a = 0$$

As a result, we see that they must be linearly independent as the only relation is the trivial one.

- $W = \text{span}(\mathcal{B})$.

  To do this, we recall that because W is defined as the span of v_1, v_2, v_3 all vectors $w \in W$ can be written as a linear combination of the three elements. That is, for each $w \in W$, there exist scalars a,b,c such that

$$w = av_1 + bv_2 + cv_3$$

  Next, we observe that our chosen basis is simply $\{v_2, v_3\}$, and that

$$v_1 = 2 * v_2 - v_3$$

  We now show this by substituting in values for v_1, v_2, v_3

$$\begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix} = 2 * \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

  Applying scalar multiplication yields

$$\begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

  Applying vector subtraction yields

$$\begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix}$$

  This is trivially true, showing the equality. If we substitute this result into the decomposition of w, we obtain.

$$w = a(2v_2 - v_3) + bv_2 + cv_3$$

  By distributivity

$$w = 2av_2 - av_3 + bv_2 + cv_3$$

  By commutativity and associativity

$$w = (2a + b)v_2 + (c - a)v_3$$

  Since $(2a + b)$ and $(c - a)$ are still scalars by closure over addition of the real field, we now know that each w can be written as a linear combination of $\mathcal{B}$ since $\mathcal{B} = \{v_1, v_2\}$. Because each w can be written as a linear combation of $\mathcal{B}$, we write that $W = \text{span}(\mathcal{B})$.

As a result, we know that $\mathcal{B}$ is a basis for W, and can now finally determine that.

$$\dim W = |\mathcal{B}| = 2$$

$\square$

2. Recall that $M_n(\mathbb{R})$ is the (real) vector space of $n \times n$ matrices with real entries. Recall that if $A \in M_n(\mathbb{R})$, then A_{ij} is the (i, j)th entry of A. Let

$$W_1 = \{ A \in M_n(\mathbb{R}) \mid A_{ij} = 0 \text{ if } i > j \}$$
$$W_2 = \{ A \in M_n(\mathbb{R}) \mid A_{ij} = 0 \text{ if } i < j \}$$

be the subspaces of upper-triangular matrices and lower-triangular matrices, respectively.[2]

(a) (4 points) Prove that $M_n(\mathbb{R}) = W_1 + W_2$.

Solution. In order to prove that $M_n(\mathbb{R}) = W_1 + W_2$, we simply need to prove that for all $m \in M_n(\mathbb{R})$, there exists $w_1 \in W_1, w_2 \in W_2$ such that $m = w_1 + w_2$. To do this, we consider arbitrary m, and let each of its elements be addressable by the form m_{ij}. We begin by noting that w_1, w_2 are matricies, and as a result we can write them with the same form. we propose the following definitions of w_1, w_2 given m

$$w_{1,ij} = \begin{cases} 0 & \text{i > j} \\ \text{m}_{ij} & \text{i} \leq j \end{cases}$$

$$w_{2,ij} = \begin{cases} 0 & \text{i} \leq j \\ \text{m}_{ij} & \text{i > j} \end{cases}$$

It is clear that both of these definitions are in the subspace, as can be seen from their first line matching the requirement to be in their respective subspaces. To show that this provides a decomposition, we simply analyze the sum for all cases.

For $i \leq j$

$$m_{ij} = w_{1,ij} + w_{2,ij}$$

By definition of w_1, w_2

$$m_{ij} = m_{ij} + 0$$

By additive identity

$$m_{ij} = m_{ij}$$

For $i > j$

$$m_{ij} = w_{1,ij} + w_{2,ij}$$

By definition of w_1, w_2

$$m_{ij} = 0 + m_{ij}$$

By additive identity

$$m_{ij} = m_{ij}$$

Since each element of m is correctly decomposed by this, m is decomposed into two vectors from W_1, W_2, meaning we have shown that each element of $M_n(\mathbb{R})$ can be decomposed into the sum of two vectors from W_1, W_2, which proves that

$$M_n(\mathbb{R}) = W_1 + W_2$$

□

(b) (4 points) Prove that $M_n(\mathbb{R})$ is *not* the direct sum of W_1 and W_2.

[2] You don't have to prove that they are subspaces, you can assume this.

Solution. To show that this is true, one has to recall that for something to be the direct sum of two other subspaces, it must fulfill two properties

- It must be the sum of the two subspaces
  This was already shown in part (A)
- The two subspaces' intersection must only be the zero element.
  This is not true. Consider the identity matrix

  $$I = \begin{bmatrix} 1 & 0 & \dots \\ \vdots & \ddots & \\ 0 & & 1 \end{bmatrix}$$

  We assert that this is in both W_1, W_2, which would mean the intersection contains I as well as the nontrivial solution. To prove this, rewrite I as the following set of components

  $$I_{ij} = \begin{cases} 0 & \text{i} \neq j \\ 1 & \text{i = j} \end{cases}$$

  We see that for all i > j, $I_{ij} = 0$, which means it is in W_1. Additionally, for all i < j, $I_{ij} = 0$, which means it is in W_2. As a result, $W_1 \cap W_2 \neq \{0\}$ And as a result, $M_n(\mathbb{R})$ is **not** the direct sum of W_1, W_2.

□

3. For each of the following statements, determine whether it is **true** or **false**. If it is true, prove it. If it is false, give an explicit counter example or an explicit reason why.

(a) (4 points) Let V be a vector space and let $W_1, W_2 \subset V$ be subspaces. If $\mathcal{B}_1$ is a basis for W_1 and $\mathcal{B}_2$ is a basis for W_2, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $W_1 + W_2$.

Solution. This is not true, consider the following definitions for each term.

$$\begin{aligned} V &:= \mathbb{R}^2 \\ W_1 &:= \operatorname{span}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}\right) \\ W_2 &:= \operatorname{span}\left(\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}\right) \end{aligned}$$

It is clear that the bases vectors for each of these subsets can be defined as following:

$$\begin{aligned} \mathcal{B}_1 &= \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \\ \mathcal{B}_2 &= \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} \end{aligned}$$

The linear indepndence of $\mathcal{B}_2$ is trivial as there is one nonzero vector, and the linear independence of $\mathcal{B}_1$ is shown in part (C) of this problem. They also clearly span their respective subsets by definition. We then observe that

$$\mathcal{B}_1 \cup \mathcal{B}_2 = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

However, we note that bases must be linearly independent, and this new subset has a nontrivial relation. In particular, if one observes

$$\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \vec{0}$$

By standard vector addition

$$\begin{pmatrix}1-1\\1-1\end{pmatrix} = 0$$

By additive inverse

$$\begin{pmatrix}0\\0\end{pmatrix} = 0$$

We now see that we have a nontrivial relation among the subset, which means it cannot be a basis. Since it isn't a basis, it can't be a basis for $W_1 + W_2$, showing the lemma is false. ☐

(b) (4 points) Let $F(\mathbb{R}, \mathbb{R})$ be the (real) vector space of functions $f : \mathbb{R} \to \mathbb{R}$. The set

$$W = \{\, f \in F(\mathbb{R}, \mathbb{R}) \mid f(x) \leq 1 \text{ for all } x \in \mathbb{R} \,\}$$

is a subspace of $F(\mathbb{R}, \mathbb{R})$.

Solution. This is not true, consider the functions

$$f(x) = 1, g(x) = 0.5$$

It is clear that $f, g \in W$ as they are functions on the real numbers that are always less than or equal to 1. However, recall that one of the properties of a subspace(or a vectorspace in general) is closure under addition. This implies that

$$(f + g)(x) \in W$$

To see if this is true, we see if (f + g)(x) meets the properties to be in the set. That is, we assert that

$$(f + g)(x) \leq 1 \forall x \in \mathbb{R}$$

Applying standard function addition yields.

$$f(x) + g(x) \leq 1 \forall x \in \mathbb{R}$$

Substituting in the values given

$$1 + 0.5 \leq 1 \forall x \in \mathbb{R}$$

$$1.5 \leq 1 \forall x \in \mathbb{R}$$

This is a clear contradiction, and as a result we conclude that W is **not** a subspace of $F(\mathbb{R}, \mathbb{R})$ □

(c) (4 points) Let V be a vector space. If $S_1, S_2 \subset V$ are (nonempty) subsets of V such that $S_2 \subset \operatorname{span}(S_1)$, then $S_1 \cup S_2$ is linearly dependent.

Solution. This is not true, consider the following definitions for each variable

$$V := \mathbb{R}^2$$

$$S_1 := \{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\}$$

$$S_2 := \{\begin{pmatrix}0\\1\end{pmatrix}\}$$

We see that $S_2 \subseteq \operatorname{span}(S_1)$ because all elements of S_2 can be decomposed into a linear combination of the elements of S_1. In particular, one can see that if

$$\begin{pmatrix}0\\1\end{pmatrix} = a\begin{pmatrix}0\\1\end{pmatrix} + b\begin{pmatrix}1\\0\end{pmatrix}$$

Substituting in $a = 1, b = 0$ reveals that

$$\begin{pmatrix}0\\1\end{pmatrix} = 1\begin{pmatrix}0\\1\end{pmatrix} + 0\begin{pmatrix}1\\0\end{pmatrix}$$

From lemma in class, we know that $0 * v = 0 \forall v \in V$. So we can write that

$$\begin{pmatrix}0\\1\end{pmatrix} = 1\begin{pmatrix}0\\1\end{pmatrix}$$

Applying multiplicative identity.

$$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

Since these are all of the elements of S_2, we conclude that it is a subset of the span. However, one should note that

$$\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\} \cup \left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$$

Is linearly independent. To show this, consider all scalars a and b such that:

$$a\begin{pmatrix}0\\1\end{pmatrix} + b\begin{pmatrix}1\\0\end{pmatrix} = 0$$

We need to show that there is no nontrivial relation for this to be linearly independent. We can do this by applying standard vector multiplication.

$$\begin{pmatrix}0a\\1a\end{pmatrix} + \begin{pmatrix}1b\\0b\end{pmatrix} = 0$$

Standard multiplication lemmas

$$\begin{pmatrix}0\\a\end{pmatrix} + \begin{pmatrix}b\\0\end{pmatrix} = 0$$

Standard vector addition with additive identity.

$$\begin{pmatrix}b\\a\end{pmatrix} = 0$$

Now, we consider each variable seperately, noting that the following equations now must hold true

$$a = 0$$
$$b = 0$$

This implies that the only solution is the trivial one, proving linear **independence** and that the lemma is **false**. □