# Midterm 1

## UCLA: Math 115A, Spring 2018

*Instructor:* Jens Eberhardt
*Date: 23 Arpil 2018*

- This exam has 4 questions, for a total of 24 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: STEVEN LA

ID number: [redacted]

| Question | Points | Score |
|---|---|---|
| 1 | 6 | 6 |
| 2 | 6 | 4 |
| 3 | 6 | 6 |
| 4 | 6 | 6 |
| Total: | 24 | 22 |

1. Prove or disprove the following statements.

(a) (3 points)

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \in \mathrm{Span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$$

Here all vectors are in $\mathbb{R}^3$.

(b) (3 points) Let

$$T : \mathbb{R}^2 \to \mathrm{P}(\mathbb{R}), \begin{pmatrix}a\\b\end{pmatrix} \mapsto (a+5b)x^2 + (-2a+3b)x^3.$$

Then

$$x^2 + 2x^3 \in \mathrm{im}(T).$$

Here $\mathrm{im}(T)$ denotes the *image* (also called *range*) of $T$.

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a.) $a\begin{pmatrix}-1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c}-1 & 1 & 0 & 1\\ 0 & -1 & 1 & 2\\ 1 & 0 & -1 & 3\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & -1 & 0 & -1\\ 0 & -1 & 1 & 2\\ 0 & 1 & -1 & 4\end{array}\right)$

$\rightsquigarrow \left(\begin{array}{ccc|c}1 & 0 & -1 & -3\\ 0 & 1 & -1 & -2\\ 0 & 0 & \boxed{0} & \boxed{6}\end{array}\right) \rightarrow 0 \neq 6$, $\Rightarrow$ no solution $\Rightarrow \nexists\ a,b,c \in \mathbb{R}$ st $a\begin{pmatrix}-1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \Rightarrow$ $\boxed{\begin{pmatrix}1\\2\\3\end{pmatrix} \text{ is } \underline{\text{not}} \text{ in the span of } \left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}}$ ✓

b.) If $x^2 + 2x^3 \in \mathrm{im}(T)$, then $\exists \begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{R}^2$, $a, b \in \mathbb{R}$ st $\begin{pmatrix}a\\b\end{pmatrix} \mapsto x^2 + 2x^3$, ie. there must be a solution to the set of eq.s:

$$(\mathrm{I})\quad \begin{aligned} a + 5b &= 1 \\ -2a + 3b &= 2 \end{aligned} \Rightarrow \begin{aligned} 2a + 10b &= 2 \\ + \; -2a + 3b &= 2 \\ \hline 13b &= 4 \end{aligned}$$

$b = \frac{4}{13}$

$a + 5\left(\frac{4}{13}\right) = 1$

$a = 1 - \frac{20}{13}$ ✓

$a = \frac{13}{13} - \frac{20}{13}$

$a = \frac{-7}{13}$

Because $a = -\frac{7}{13}$, $b = \frac{4}{13}$ and $a, b \in \mathbb{R}$, $\exists$ a sol. to the set of eq.s (I). ✓

$\Rightarrow \boxed{x^2 + 2x^3 \in \mathrm{im}(T)}$

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$0 \quad 0 \quad -1 \mid 0 \qquad -1 \quad 0 \quad -1 \mid 0$



$0 \quad -1 \quad -1 \mid 0$

2. (a) (4 points) Let

$$S = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

Prove or disprove that S is a basis of $\mathbb{R}^3$.

(b) (2 points) Is $x = (1.3245, 10^{420}, \pi) \in \text{Span}(S)$? (*Hint*: Do not solve a system of linear equations here)

col. vec.

a.) Because $\dim(\mathbb{R}^3) = 3$, it is sufficient to show that S is lin. indp. in order to prove that S is basis of $\mathbb{R}^3$. ✓

$$\Rightarrow a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\1&1&0&0\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&1&-1&0\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&1&0\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right) \Rightarrow \boxed{a=b=c=0}$$ ✓

Hence, since only the trivial solution exists to the lin. comb of vec.s in S equaling zero, S is lin. indp $\Rightarrow$ $\boxed{\text{S is a basis of } \mathbb{R}^3}$ 4

b.) $x = \begin{pmatrix}1.3245\\10^{420}\\\pi\end{pmatrix} \notin \text{span}(S)$ because S is defined over the $\mathbb{R}$-field. x contains $\pi \notin \mathbb{R}$, so $\boxed{x \notin \text{span}(S).}$

$\pi$ is a real number

0

~~*Solution*: Because $x \in \mathbb{R}^3$ and S is a basis of $\mathbb{R}^3$ (by part A) $\Rightarrow$ S generates $\mathbb{R}^3 \Rightarrow$ span(S) $= \mathbb{R}^3 \Rightarrow \exists\, u_1, u_2, u_3 \in$ S and $a_1, a_2, a_3 \in \mathbb{R}$ st~~

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$x = a_1 u_1 + a_2 u_2 + a_2 u_3 \Rightarrow x \in \text{span}(S)$.

3. Prove or disprove (by giving a counterexample) the following statements.

(a) (4 points) Let $T, S : V \to W$ be linear transformations between two vector spaces $V, W$ over a field $F$. Then
$$U = \{w \in W \mid \text{There is a } v \in V \text{ such that } T(v) = S(v) = w\}$$
is a subspace of $W$.

(b) (2 points) The following set $S$ is a subspace of $\mathbb{R}^2$:
$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 | (x+1)^2 + y^2 = 1 \right\}.$$

a.) i.) 0 ∈ Subspaces: Because V is a vector space, $\exists\, 0 \in V$ by V.S Axiom 3. So,

$$\left.\begin{aligned} T(\underset{V}{0}) &= T(\underset{F}{0}\cdot\underset{V}{0}) = \underset{F}{0}\underbrace{T(0)}_{W} = \underset{W}{0} \\ S(\underset{V}{0}) &= S(\underset{F}{0}\cdot\underset{V}{0}) = \underset{F}{0}\underbrace{S(0)}_{W} = \underset{W}{0} \end{aligned}\right\} \text{ Hence, } T(0)=S(0) = \underset{W}{0}.$$

$\Rightarrow 0 \in U.$ //

ii.) Closed under addition: Let $x, y \in U$. Then, $\exists\, a, b \in V$

st. $T(a) = S(a) = x$
$T(b) = S(b) = y$ $\Rightarrow$ $T(a) + T(b) = S(a) + S(b) = (x+y)$ Nice!
$T(a+b) = S(a+b) = (x+y)$

Because $(a+b) \in V$ and $T(a+b) = S(a+b) = \underset{W}{(x+y)} \Rightarrow (x+y) \in U$ //

iii.) Closed under scalar multiplication: Let $x \in V$, $c \in F$. Then, $\exists\, a \in V$ st

$T(a) = S(a) = x. \Rightarrow$ $cT(a) = cS(a) = cx$
$T(ca) = S(ca) = cx$

Because $(ca) \in V$ and $T(ca) = S(ca) = \underset{W}{cx} \Rightarrow cx \in U$ //.

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Therefore, since U satisfies the 3 properties of subspace, $\boxed{U \text{ is a subspace of } W.}$

b.) Counter-example: $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in S$ $(-1+1)^2 + 1^2 = 1$

$$3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} \to (-3+1)^2 + 3^2 \overset{?}{=} 1$$
$$(-2)^2 + 9 \overset{?}{=} 1$$
$$4 + 9 = 13 \neq 1.$$

So, S is not closed under scalar multiplication

$\Rightarrow$ $\boxed{S \text{ is not a subspace of } \mathbb{R}^2.}$

~~2~~ 0

$\begin{array}{r} 18 \\ \times 2 \\ \hline 36 \end{array}$ $8-8=0$ $10-10=0$ $\begin{array}{r} 36 \\ +24 \\ \hline 60 \end{array}$



4. Let $F$ be a field. For $A \in \mathrm{M}_{2,2}(F)$ we define the *trace* of $A$, denoted $\mathrm{tr}(A)$, by

$5\cdot 3$

$$\mathrm{tr}(A) = \mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

and the *determinant* of $A$, denoted $\det(A)$, by

$4\cdot 2 - 8\cdot 1$

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Prove or disprove that the following subsets are subspaces of of $\mathrm{M}_{2,2}(F)$.

(a) (4 points)
$$W = \{A \in \mathrm{M}_{2,2}(F) \mid \mathrm{tr}(A) = 0\}$$

(b) (2 points)
$$W = \{A \in \mathrm{M}_{2,2}(F) \mid \det(A) = 0\}$$

a.) i.) 0 ∈ subspaces: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$, since $\mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0+0 = 0$ //

ii.) Closed addition: Let $X, Y \in M_{2,2}(F)$. Then $\mathrm{tr}\begin{pmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{pmatrix} = 0$ and $\mathrm{tr}\begin{pmatrix} Y_{11} & Y_{21} \\ Y_{12} & Y_{22} \end{pmatrix} = 0 \Rightarrow \begin{array}{l} X_{11}+X_{22}=0 \\ Y_{11}+Y_{22}=0 \end{array} \Rightarrow (X_{11}+Y_{11})+(X_{22}+Y_{22})=0$. ✓

This sum corresponds directly to $\mathrm{tr}(X+Y)$. Hence $\mathrm{tr}(X+Y)=0$
$\Rightarrow (X+Y) \in W$ //

iii.) Closed multiplication: Let $X \in M_{2,2}(F)$. Then $\mathrm{tr}\begin{pmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{pmatrix} = 0$.
$\Rightarrow X_{11}+X_{22}=0$. Let $c \in F$. Then $cX = \begin{pmatrix} cX_{11} & cX_{21} \\ cX_{12} & cX_{22} \end{pmatrix}$ and
$\mathrm{tr}(cX) = cX_{11}+cX_{22} = c(X_{11}+Y_{22}) = c(0) = 0$. ✓
So, $cX \in W$ //.

Nice!

Therefore, $\boxed{W \text{ is a subspace.}}$ ■ 4

b.) Counter-Example: $\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 10 & 10 \end{pmatrix} \in W$

$\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 10 & 10 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 18 & 12 \end{pmatrix} \rightarrow 5(12) - (18)2 = \det\left(\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 10 & 10 \end{pmatrix}\right)$
$60 - 36 = 24 \neq 0$ ✓ 2

Hence, $\left(\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 10 & 10 \end{pmatrix}\right) \notin W \Rightarrow W$ is not closed under addition, $\boxed{W \text{ is not a subspace.}}$ 6