Math 115A
Linear Algebra

Midterm 2

February 25th, 2019

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator or any electronic device is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; indicate if you use srap paper. Don't write on the back of the pages.

Do not forget to write your name and UID in the space below.

Do not engage in any kind of academic dishonesty, including looking at someone else's exam or letting someone else look at your exam.. Remember that you are bound by a conduct code!

Name: ______________________________

Student ID number: ______________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Problem 1.

Indicate whether the following statements are either true or false. Circle your answer.

(a) [1pts.] As a vector space over $\mathbb{R}$, $M_{n\times n}\mathbb{C}$ is isomorphic to $\mathbb{R}^{4n^2}$.

T **F**

(b) [1pts.] If $T : V \to W$ is a linear map and V has two finite ordered bases α and β, and W has two finite ordered bases γ and δ, then $[T]_\beta^\delta = [id]_\delta^\gamma [T]_\alpha^\gamma [id]_\beta^\alpha$

T **F**

(c) [1pts.] Every diagonalizable linear map is also invertible.

T **F**

(d) [1pts.] If β is a basis for V and $T : V \to W$ a linear map, and $T(\beta)$ is a basis for W, then T is an isomorphism.

T **F**

(e) [1pts.] If $A, B \in M_{n\times n}F$ and $\det A = \det B$, then A and B are similar.

T **F**

(f) [1pts.] If x is an eigenvector for the matrix A^2 then it is also an eigenvector for the matrix A.

T **F**

(g) [1pts.] Suppose that U and T are linear operators on a finite dimensional vector space, and that $U \circ T$ is invertible. Then U and T are each invertible.

T **F**

(h) [1pts.] If A is a 3×3 matrix with columns $[x_1 x_2 x_3]$, and B has columns $[x_3 x_1 x_2]$ then $\det B = \det A$.

T **F**

(i) [1pts.] Suppose $T : V \to W$ and $U : W \to Z$ are linear maps and that $U \circ T : V \to Z$ is invertible. Then U and T are each invertible.

T **F**

(j) [1pts.] The map $\det : M_{n\times n}F \to F$ is linear.

T **F**

Problem 2.

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -13x + 10y \\ -21x + 16y \end{bmatrix}$.

(a) [6pts.] Compute the matrix representation of T in the ordered basis $\left\{\begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$.

$T = \begin{bmatrix} -13 & 10 \\ -21 & 16 \end{bmatrix}$

$\begin{array}{r} -21 \\ \times 5 \\ \hline -105 \end{array}$ $\begin{array}{r} 16 \\ \times 7 \\ \hline 112 \end{array}$

$T\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ $T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$[T]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) [4pts.] Is there a basis β for $\mathbb{R}^2$ so that $[T]_\beta = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$? Be sure to justify your answer.

$[T]_\beta = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$\begin{pmatrix} -13 \\ -21 \end{pmatrix} = 2\beta_1 + 3\beta_2$

Problem 3.

For $A \in M_{n\times n}F$, recall that the trace of A is $\sum_{i=1}^{n} A_{i,i}$.

(a) [5pts.] Show that for $A, B \in M_{n\times n}F$ the trace of AB is equal to the trace of BA. Feel free to use anything we've talked about so far, unless it is essentially exactly this statement.

$AB_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$ for $1 \le i \le n$ $1 \le j \le n$

Then $tr(AB) = \sum_{l=1}^{n} AB_{l,l} = \sum_{l=1}^{n} \sum_{k=1}^{n} A_{lk} B_{kl}$

$BA_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}$ for $1 \le i \le n$ $1 \le j \le n$

Then $tr(BA) = \sum_{l=1}^{n} BA_{l,l} = \sum_{l=1}^{n} \sum_{k=1}^{n} B_{lk} A_{kl}$

$= \sum_{l=1}^{n} \sum_{k=1}^{n} A_{kl} B_{lk}$

$= \sum_{k=1}^{n} \sum_{l=1}^{n} A_{kl} B_{lk}$

$tr(AB)$ and $tr(BA)$ are summations of the same form

So, $tr(AB) = tr(BA)$.

(b) [5pts.] Show that if A and B are similar, then the trace of A is equal to the trace of B. Feel free to use anything we've talked about so far, unless it is essentially exactly this statement.

Let A, B be similar, $A, B \in M_{n\times n} F$

Then $\exists Q$ invertible st $B = Q^{-1}AQ$

$tr(B) = tr(Q^{-1}AQ)$

$tr(B) = tr(Q^{-1}(AQ))$

$tr(B) = tr((AQ)Q^{-1})$ by above

$tr(B) = tr(AQQ^{-1})$

$tr(B) = tr(AI)$

$tr(B) = tr(A)$ $\boxtimes$

Problem 4.

(a) [6pts.] If $T : V \to V$ is a linear operator on an n-dimensional F-vector space and T is diagonalizable, show that $\chi_T(t) = (c_1 - t)(c_2 - t)\ldots(c_n - t)$, where the c_i are scalars in F. Feel free to use anything we've talked about so far, unless it is essentially exactly this statement.

Suppose $c_1 \ldots c_n$ are eigen values of T.

Then $c_1 \ldots c_n \in F$ st $\chi_T(c_i) = 0$ for $1 \le i \le n$

($c_1 \ldots c_n$ are roots of $\chi_T(t)$)

If $c_1 \ldots c_n$ are roots of $\chi_T(t)$

Then $(c_i - t)$ must be a factor of $\chi_T(t)$

So, $\chi_T(t)$ must be a product of $(c_i - t)$ for all $1 \le i \le n$

Thus, $\chi_T(t) = (c_1 - t)(c_2 - t) \ldots (c_n - t)$. where $c_1 \ldots c_n \in F$.

(b) [4pts.] Use the previous question to show that the linear map $T : P_2\mathbb{R} \to P_2\mathbb{R}$ defined by $T(a + bx + cx^2) = (3b - c) + (2bx) + (a + 7b)x^2$ is not diagonalizable.

$$T = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 2 & 0 \\ 1 & 7 & 0 \end{bmatrix} \qquad \chi_T(t) = \det\begin{pmatrix} -t & 3 & -1 \\ 0 & 2-t & 0 \\ 1 & 7 & -t \end{pmatrix}$$

$$\chi_T(t) = -t(2-t)(-t) - 3(0) - 1(t-2)$$
$$= 2t^2 - t^3 - t + 2$$
$$= -t^3 + 2t^2 - t + 2$$
$$= 2(-t+1)(t^2+1)$$
$$= 2(1-t)(i-t)(-i-t)$$

$i \notin \mathbb{R}$ so T is not diagonizable