Math 115A
Linear Algebra

Final

March 18th, 2019

Instructions: You have 3 hours to complete this exam. There are 8 questions, worth a total of 80 points. This test is closed book and closed notes. No calculator or any electronic device is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; do no go over the page, indicate if you use scrap paper.

Do not forget to write your name and UID in the space below.

Do not engage in any kind of academic dishonesty, including looking at someone else's exam or letting someone else look at your exam. Remember that you are bound by a conduct code!

Name: ______________________________

Student ID number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Problem 1.

Indicate whether the following statements are either true or false. Circle your answer.

(a) [1pts.] If $A, B, C \in M_{n\times n}F$ and $A = C^{-1}BC$ then every eigenvector of A is an eigenvector of B, and vice versa.

T **F**

(b) [1pts.] If $T : V \to V$ is a linear operator on a finite dimensional complex vector space, then T has an eigenvalue.

T **F**

(c) [1pts.] If $\{x, y\}$ is linearly independent, so is $\{x + y, x\}$.

T **F**

(d) [1pts.] If $T : V \to W$ is a linear map between finite dimensional spaces, then there are ordered bases α for V, β for W so that $([T]_\alpha^\beta)_{i,j}$ is zero or one for $i = j$, and zero for $i \neq j$.

T **F**

(e) [1pts.] If $A, B \in M_{n\times n}F$ and $\chi_A(t) = \chi_B(t)$, then A and B are similar.

T **F**

(f) [1pts.] If T, U are diagonalizable linear operators on a finite dimensional vector space V, and $TU = UT$, then every eigenvector for T is an eigenvector for U.

T **F**

(g) [1pts.] If $T : V \to V$ is an operator on a finite dimensional vector space and $\operatorname{rank} T = \operatorname{rank} T^*$, then $T = T^*$.

T **F**

(h) [1pts.] If $W \subset M_{2\times 2}\mathbb{C}$ is a 1 dimensional subspace, then $W^\perp$ is 1 dimensional.

T **F**

(i) [1pts.] If $T : V \to V$ is an operator on a finite dimensional inner product space and for some $y \in V$, $\langle (T - \lambda)(x), y\rangle = 0$, then x is an eigenvector for T with eigenvalue λ.

T **F**

(j) [1pts.] The real numbers are finite dimensional as a vector space over the rational numbers.

T **F**

Problem 2.
For V a vector space and W a subspace of V and $v \in V$, $v + W = \{x \in V : x = v + w \text{ for some } w \in W\}$.

(a) [5pts.] Show that $v + W$ is a subspace of V if and only if $v \in W$.

$W \subseteq V$

suppose $v \notin W$

then $\forall\ w_1, w_2 \in W$, $(v+w_1), (v+w_2) \in v+W$

$(v+w_1) + (v+w_2) = 2v + (w_1 + w_2) \notin v+W$

So $v+W$ is not a subspace

By contraposition, $v+W$ is a subspace of $V \Rightarrow v \in W$

If $v \in W$

then $\forall\ w_1, w_2 \in W$ $(v+w_1), (v+w_2) \in v+W$

$(v+w_1) + (v+w_2) = v + (w_1 + v + w_2)$

$(w_1 + v + w_2) \in W$ so $v + (w_1 + v + w_2) \in v+W$

$\forall c \in F$

$c(v+w_1) = cv + cw_1 = v + (cw_1 + (c-1)v)$

$(cw_1 + (c-1)v) \in W$ so $v + (cw_1 + (c-1)v) \in v+W$

Since $v \in W$, choose $w_1 = -v$

then $v + w_1 = v - v = 0 \in v+W$

So, if $v \in W$, $v+W$ is a subspace of V.

$v+W$ is a subspace of $V \Leftrightarrow v \in W$ ☒

(b) [5pts.] Show that $x + W = y + W$ if and only if $x - y \in W$.

If $x + W = y + W$

Then $\forall z \in x+W \quad z \in y+W$

$z = x + w_1 = y + w_2 \qquad x, y \in V \qquad w_1, w_2 \in W$

$x - y = w_2 - w_1$

Since W is a subspace of V

$(w_2 - w_1) \in W$

So $(x-y) \in W$

If $x - y \in W \quad \forall x, y \in V$

Let $x - y = w_1 \in W$

Let $w_1 = w_2 - w_3 \quad \forall w_2, w_3 \in W$.

$x - y = w_2 - w_3$

$x + w_3 = y + w_2$

Then $\forall z_1 \in x+W$, $z_1 \in y+W$ and $\forall z_2 \in y+W$, $z_2 \in x+W$

So $x + W = y + W$

$x + W = y + W \iff x - y \in W$ $\boxtimes$

Problem 3.

Consider $\mathbb{R}^2$ with a function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $f((x, y), (u, v)) = u(2x + y) + v(x + 2y)$.

(a) [4pts.] Show that f is an inner product.

for $(x,y),(u,v),(s,t) \in \mathbb{R}^2$ $\quad c \in \mathbb{R}$

$f((x,y)+(u,v),(s,t)) = f((x+u,y+v),(s,t))$

$= s(2(x+u)+(y+v)) + t((x+u)+2(y+v))$

$= s(2x+y) + s(2u+v) + t(x+2y) + t(u+2v)$

$= s(2x+y) + t(x+2y) + s(2u+v) + t(u+2v)$

$= f((x,y),(s,t)) + f((u,v),(s,t))$ ✓

$\overline{\langle x,y\rangle} = \langle y,x\rangle$

$f(c(x,y),(u,v)) = f((cx,cy),(u,v))$

$= u(2cx+cy) + v(cx+2cy)$

$= c[u(2x+y) + v(x+2y)]$

$= 2 f((x,y),(u,v))$ ✓

$f((x,y),(u,v)) = u(2x+y) + v(x+2y)$

$= 2ux + uy + vx + 2vy$

$= x(2u+v) + y(u+2v)$

$= f((u,v),(x,y))$ ✓

$f((x,y),(x,y)) = x(2x+y) + y(x+2y)$

$= 2x^2 + 2xy + 2y^2$

$= x^2 + (x+y)^2 + y^2 > 0$ ✓

$$-\tfrac{1}{2}(\cancel{\;}+1) + 1\left(-\tfrac{1}{2}+2\right) = \tfrac{3}{2}$$
$$f = u(2x+y) + v(x+2y)$$

(b) [3pts.] Find an orthonormal basis for $\mathbb{R}^2$ with respect to the inner product from the previous part of this question (not with respect to the standard inner product!).

$S = \{(1,0),(0,1)\}$

$b_1' = (1,0)$

$b_2' = (0,1) - \dfrac{\langle (0,1),(1,0)\rangle}{\langle (1,0),(1,0)\rangle}(1,0) = (0,1) - \dfrac{(1+0)}{2}(1,0) = \left(-\tfrac{1}{2}, 1\right)$

$b_1 = \dfrac{(1,0)}{\|(1,0)\|} = \dfrac{1}{\sqrt{2}}(1,0)$

$b_2 = \dfrac{\left(-\frac{1}{2},1\right)}{\left\|\left(-\frac{1}{2},1\right)\right\|} = \sqrt{\dfrac{2}{3}}\left(-\tfrac{1}{2},1\right)$

$\beta = \left\{\dfrac{1}{\sqrt{2}}(1,0),\ \sqrt{\dfrac{2}{3}}\left(-\tfrac{1}{2},1\right)\right\}$

(c) [3pts.] Describe the adjoint (with respect to the inner product of the previous part of this problem, not the standard inner product) of the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x,y)) = (2x+3y, 2x-3y)$. In other words, $T^*(u,v) = (w,z)$, what are w and z in terms of u and v?

$\langle T(x,y),(u,v)\rangle = \langle (x,y),(w,z)\rangle$

$\langle (2x+3y, 2x-3y),(u,v)\rangle = \langle (x,y),(w,z)\rangle$

$u(4x+6y+2x-3y) + v(2x+3y+4x-6y) = w(2x+y) + z(x+2y)$

$u(6x+3y) + v(6x-3y) = w(2x+y) + z(x+2y)$

$3u(2x+y) + 3v(2x-y) = w(2x+y) + z(x+2y)$

let $x=1 \quad y=-2$

$3v(4) = z(-3)$

$-4v = z$

let $x=2 \quad y=-1$

$9u + 3v(5) = w(3) \qquad w = 3u+5v$

$\boxed{\begin{array}{l} w = 3u+5v \\ z = -4v \end{array}}$

Problem 4.

Let $T: V \to W$ be a linear map, where V and W are finite dimensional vector spaces.

(a) [2pts.] Show that if T is one-to-one, then V is isomorphic to $R(T)$.

$\dim V = \dim R(T) = \text{rank} T$

If T 1-1

Then $\text{rank} T = \dim V$

$\text{rank} T = \dim(R(T))$

So $\dim V = \dim(R(T))$ and V is isomorphic to $R(T)$ $\boxtimes$

(b) [2pts.] Show that if T is one-to-one, then if β is a basis for V, $T(\beta)$ is a basis for $R(T)$.

T 1-1

β is basis for V

Then β is linearly independent and

$T(\beta)$ forms a linearly independent set.

$* \; T(\beta) = \{x \in W \mid x = T(b_i) \; b_i \in \beta\}$

$\dim V = \dim \text{span} \beta = \dim \text{span} T(\beta) \Rightarrow \dim R(T)$

Since $T(\beta)$ is a linearly independent set with

$\dim \text{span}(T(\beta)) = \dim R(T)$

$T(\beta)$ forms a basis for $R(T)$ $\boxtimes$

(c) [4pts.] Show that if T is one-to-one, then there is a linear map $U : W \to V$ so that $U \circ T = I_V$.

If T is 1-1

T is invertible on $R(T) \subseteq W$

So for $T: V \to R(T)$, $R(T) \subseteq W$ T is bijective

There exists a $T^{-1}: R(T) \to V$ st $T^{-1} \circ T = I_V$

So there exists a linear map $U: W \to V$

st $U \circ T = I_V$ ☒

(d) [2pts.] For the linear map U from the previous part of this question, if $V = M_{2\times2}F$ and $W = P_6F$, what is $\dim N(U)$?

$\dim W = \text{null}\, U + \text{rank}\, U$ U maps ONTO V

$7 = \text{null}\, U + \text{rank}\, U$

$7 = \text{null}\, U + \dim V$

$7 = \text{null}\, U + 4$

$\boxed{3 = \text{null}\, U}$

Problem 5.

Consider the linear map $T : M_{2\times 2}\mathbb{R} \to M_{2\times 2}\mathbb{R}$ defined by $T(A) = A^T$.

(a) [4pts.] What are the eigenvalues of T?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}$$

$$\begin{aligned} \det T - \lambda I &= (1-\lambda)(-\lambda(-\lambda(1-\lambda)) - 1(1(1-\lambda))) \\ &= (1-\lambda)(\lambda^2 - \lambda^3 + \lambda - 1) \\ &= (1-\lambda)(-\lambda^3 + \lambda^2 + \lambda - 1) \\ &= (1-\lambda)(1-\lambda)(\lambda^2 - 1) \\ &= (1-\lambda)(1-\lambda)(\lambda+1)(\lambda-1) \\ &= -(1-\lambda)^3(1+\lambda) \end{aligned}$$

$$\begin{array}{r|rrrr} 1 & -1 & 1 & 1 & -1 \\ & \downarrow & -1 & 0 & 1 \\ \hline & -1 & 0 & 1 & 0 \end{array}$$

eigenvalues are 1 and -1

(b) [4pts.] Find a basis of $M_{2\times 2}\mathbb{R}$ consisting of eigenvectors for T.

$\lambda_1 = 1$

$T - \lambda I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$(T - \lambda_1 I)(x) = 0$

$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$x = s\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + u\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$s, t, u \in \mathbb{R}$

$\lambda_2 = -1$

$(T - \lambda_2 I) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$(T - \lambda_2 I)(x) = 0$

$\left(\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

$x = s\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad s \in \mathbb{R}$

$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

(c) [2pts.] Is T diagonalizable? Be sure to justify your answer.

Yes. The algebraic multiplicity of each of the eigen values matches their geometric multiplicity.

$\lambda_1 = 1$ has alg mult $= 3$ ✓

dim $N(T - \lambda_1 I) = 3$

$\lambda_2 = -1$ has alg mult $= 1$ ✓

dim $N(T - \lambda_2 I) = 1$

Problem 6.

(a) [5pts.] Suppose that $T : V \to V$ is a self adjoint operator (self adjoint means that $T = T^*$) on a finite dimensional inner product space. Show that all the eigenvalues of T are real. Feel free to use any results from the class or the text, unless it is essentially or exactly this.

Suppose x is an eigenvector of T corresponding to λ

$\langle T(x), x\rangle = \langle x, T^*(x)\rangle$

$\langle \lambda x, x\rangle = \langle x, T(x)\rangle$

$\langle \lambda x, x\rangle = \langle x, \lambda x\rangle$

$\lambda\langle x, x\rangle = \bar{\lambda}\langle x, x\rangle$

$\lambda = \bar{\lambda}$

then $\lambda \in \mathbb{R}$

This holds for all eigen values of T. $\boxtimes$

(b) [5pts.] Suppose that $T : V \to V$ is a self adjoint operator (self adjoint means that $T = T^*$) on a finite dimensional inner product space. Suppose also that for all nonzero $x \in V$, $\langle T(x), x\rangle > 0$. Define a new function $\langle -, -\rangle_N : V \times V \to V$ by $\langle x, y\rangle_N = \langle T(x), y\rangle$. Show that $\langle -, -\rangle_N$ is an inner product.

$\forall x, y, z \in V \quad \forall c \in \mathbb{R}$

$$\begin{aligned}\langle x+z, y\rangle_N &= \langle T(x+z), y\rangle \\ &= \langle T(x)+T(z), y\rangle \\ &= \langle T(x), y\rangle + \langle T(z), y\rangle \\ &= \langle x, y\rangle_N + \langle z, y\rangle_N \quad \checkmark\end{aligned}$$

$$\begin{aligned}\langle cx, y\rangle_N &= \langle T(cx), y\rangle \\ &= \langle cT(x), y\rangle \\ &= c\langle T(x), y\rangle \\ &= c\langle x, y\rangle_N \quad \checkmark\end{aligned}$$

$$\begin{aligned}\overline{\langle x, y\rangle_N} = \overline{\langle T(x), y\rangle} &= \overline{\langle x, T^*(y)\rangle} \\ &= \overline{\langle x, T(y)\rangle} \\ &= \langle T(y), x\rangle \\ &= \langle y, x\rangle_N \quad \checkmark\end{aligned}$$

$\langle x, x\rangle_N = \langle T(x), x\rangle > 0 \quad \forall x \neq 0 \quad \checkmark$

Problem 7.

Consider the set $W = \{f \in C^\infty\mathbb{R} : f(0) = 0)\}$ as a subset of $C^\infty\mathbb{R}$ with the standard vector space structure.

(a) [4pts.] Show that W is a subspace.

$\forall f, g \in W \quad \forall c \in \mathbb{R}$

$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$ so closed under addition

$(cf)(0) = cf(0) = c \cdot 0 = 0$ so closed under scalar mult

$0(0) = 0$ so $0 \in W$

So W is a subspace $\boxtimes$

(b) [3pts.] Show that $\{e^x - 1, e^{2x} - 1\}$ is a linearly independent subset of W.

Let $a_1, a_2 \in \mathbb{R}$

Suppose $\{e^x-1, e^{2x}-1\}$ is linearly dependent

$a_1(e^x-1) + a_2(e^{2x}-1) = 0 \quad \exists a_i \neq 0 \quad 1 \leq i \leq 2$

$a_1 e^x - a_1 + a_2 e^{2x} - a_2 = 0e^x + 0e^{2x} + 0$

$a_1 e^x + a_2 e^{2x} - (a_1 + a_2) = 0e^x + 0e^{2x} + 0$

Clearly, $a_1 = 0$ and $a_2 = 0$

This contradicts the assumption that the set is linearly dependent.

So, $\{e^x-1, e^{2x}-1\}$ is linearly independent $\boxtimes$

$W = \{f \in C^\infty\mathbb{R} : f(0) = 0\}$

(c) [3pts.] Find a vector space V and a linear map $T : C^\infty\mathbb{R} \to V$ so that $N(T) = W$. Be sure to prove that your map is linear.

$V \subseteq \mathbb{R}$ $\quad T(f) = f(0)$

$N(T) = \{f \in C^\infty\mathbb{R} : f(0) = 0\} = W$

for $f, g \in C^\infty\mathbb{R}$ $\quad c \in \mathbb{R}$

$T(f+g) = (f+g)(0) = f(0) + g(0) = T(f) + T(g)$

$T(cf) = (cf)(0) = c\,f(0) = c\,T(f)$

$T(0) = 0(0) = 0$

Problem 8.

Consider the map $T : P_1\mathbb{R} \to P_1\mathbb{R}$ defined by $T(f(x)) = xf'(x) + 2f(1)$.

(a) [4pts.] What is the matrix representation of T in the standard basis.

$T(1) = 0 + 2 = 2$

$T(x) = x + 2$

$[T]_S = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$

(b) [4pts.] Find a basis β of $P_1\mathbb{R}$ so that $[T]_\beta$ is diagonal.

$\det(T - xI) = (2-x)(1-x)$

$x = 2, 1$

$\lambda_1 = 1$

$T - \lambda I = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad x = s\begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad s \in \mathbb{R}$

$(T - \lambda_1 I)(x) = 0$

$\lambda_2 = 2$

$T - \lambda_2 I = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$ $\quad x = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad s \in \mathbb{R}$

$(T - \lambda_2 I)(x) = 0$

$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

(c) [2pts.] If $[x]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, what is x?

$[x]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$x = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$x = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$