Math 115A Final Exam

TOTAL POINTS

110 / 120

QUESTION 1

Problem 1 10 pts

1.1 Part (a) 5 / 5

✓ **- 0 pts Correct**

- **- 1 pts** missing scalar multiplication
- **- 1 pts** missing zero vector
- **- 1 pts** missing addition

1.2 Part (b) 5 / 5

✓ **- 0 pts Correct**

- **- 1 pts** Imprecise
- **- 4 pts** not a counterexample

QUESTION 2

2 Problem 2 8 / 10

- **- 0 pts** Correct

✓ **- 2 pts incomplete explanation**

- **- 1 pts** incorrect definition of linear dependence
- **- 1 pts** division by zero
- **- 2 pts** wrong conclusion
- **- 1 pts** work not shown

1 proof?

QUESTION 3

3 Problem 3 10 / 10

+ 10 Point adjustment

Good!

QUESTION 4

Problem 4 10 pts

4.1 Part (a) 3 / 3

+ 3 Point adjustment

Good!

4.2 Part (b) 6 / 7

+ 6 Point adjustment

Third column has two entries swapped, but everything else is good.

2 third column should be $$$\begin{bmatrix} -1 \\ 7 \\ 0 \end{bmatrix}$$$

QUESTION 5

5 Problem 5 10 / 10

✓ **- 0 pts Correct**

- **- 2 pts** Mistake in the inverse
- **- 3 pts** work not shown
- **- 3 pts** mistake in the change of basis formula

QUESTION 6

Problem 6 10 pts

6.1 Part (a) 3 / 5

- **- 0 pts** Correct

✓ **- 2 pts not checking S(v) is non zero**

6.2 Part (b) 3 / 5

- **- 0 pts** Correct
- **- 1 pts** not justifiying v non zero
- **- 2 pts** confusion on algebraic multiplicity
- **- 4 pts** no proof

- 2 Point adjustment

3 unclear, how did you get that?

QUESTION 7

7 Problem 7 10 / 10

✓ **- 0 pts Correct**

- **- 2 pts** not justifying why diagonalizable
- **- 2 pts** work not shown for characteristic polynomial

- **2 pts** work not shown for eigenbasis

QUESTION 8

8 Problem 8 **9 / 10**

+ **0 pts** Blank

+ **9 Point adjustment**

Minor mistake in the (=>) direction.

4 this is not an if and only if implication

QUESTION 9

9 Problem 9 **9 / 10**

+ **0 pts** Blank

+ **9 Point adjustment**

Unsimplified answer.

5 simplify

QUESTION 10

10 Problem 10 **10 / 10**

+ **0 pts** Blank

+ **10 Point adjustment**

QUESTION 11

Problem 11 10 pts

11.1 Part (a) **3 / 4**

- **0 pts** Correct

- **1 Point adjustment**

6 ??

11.2 Part (b) **2 / 2**

✓ - **0 pts Correct**

11.3 Part (c) **4 / 4**

✓ - **0 pts Correct**

QUESTION 12

Problem 12 10 pts

12.1 Part (a) **5 / 5**

+ **0 pts** Blank

+ **5 Point adjustment**

12.2 Part (b) **5 / 5**

+ **0 pts** Blank

+ **5 Point adjustment**

Name: ____________________

UID: ____________________

Signature: ____________________

Instructions: You have 24 hours to complete this exam, from 12:00 AM to 11:59 PM (Pacific Daylight Time) on Tuesday, March 17, 2020. There are 12 problems worth a total of 120 points. This exam is open book and open notes. You must justify your answers and show all of your work to receive full credit. Simplify your answers as much as possible. You may lose points for answers that are not simplified. Write your solutions in the space indicated on the exam template. If your answer continues onto another page, write an easily visible note under the original question. You may use the last three pages of the exam template for scratch work. If you do not want something you write on the exam template to be graded, you must clearly cross it out. You must scan your completed exam template and upload your solutions to Gradescope by 11:59 PM (Pacific Daylight Time) on Tuesday, March 17, 2020.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	

Question	Points	Score
8	10	
9	10	
10	10	
11	10	
12	10	
Total:	120	

1. (a) (5 points)

$0 \in W_1$

Since we know in order for any $f \in W_1$ $f(2) = 0$,
0 is the polynomial that sends every x to 0 $0(x) = 0$. so
$0(2) = 0$ thus $0 \in W_1$

If $f, g \in W_1$, then $(f+g)(2) = f(2) + g(2) = 0 + 0$
Thus $f+g \in W_1$

If $f \in W_1$ and $c \in \mathbb{R}$, then $c \cdot f(2) = c \cdot 0 = 0$
Thus $cf \in W_1$

Thus W_1 is a subspace of V.

(b) (5 points)

Since $0_{P_3(\mathbb{R})}$ is the polynomial that sends all x to the real number 0,
$0_{P_3(\mathbb{R})}(2) = 0$.
Thus $0 \notin W_2$ and W_2 does not fulfill a subspace criterra.

2. (10 points)

To prove linear independence, we need to show for arbitrary $\lambda_1, \lambda_2 \in \mathbb{R}$, s.t. $\lambda_1 e^t + \lambda_2 e^{-t} = 0$, $\lambda_1 = 0$ and $\lambda_2 = 0$.

Given $\lambda_1 e^t + \lambda_2 e^{-t} = 0$ for all $t \in \mathbb{R}$

$$e^t(\lambda_1 + \lambda_2 e^{-2t}) = 0$$

$$(*) \quad \lambda_1 + \lambda_2 e^{-2t} = 0$$

for $(*)$ to be 0 for all t,

$$\lambda_1 = 0 \text{ and } \lambda_2 = 0$$

Thus the set $\{e^t, e^{-t}\}$ is linearly independent

3. (10 points)

For any $A \in M_{3\times 3}(\mathbb{R})$ where $A^t = -A$,
it has the form

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ where } A^t = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{pmatrix}$$

so $a = -a$ and $b = -d$
$e = -e$ $\quad g = -c$
$i = -i$ $\quad h = -f$

so it can be simplified

$$A = \begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & -f & 0 \end{pmatrix}$$

so $\text{span}\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\} = V$

Thus we need to prove that ~~bas~~ set $\beta = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$ is linearly independent

For $c_1, c_2, c_3 \in \mathbb{R}$,

$$c_1 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = 0$$

implies that $c_1 = c_2 = c_3 = 0$ Thus β is linearly independent

Thus β is a basis for V and $\boxed{\dim V = |\beta| = 3}$ $\square$

ax^3+bx^2+cx+d

ex^3

*Assuming that differentiation is linear

4. (a) (3 points)

To prove that T is linear,

Let $f, g \in P_3$

$$T((f+g)(x)) = (f+g)''(x) + (f+g)'(x) + (f+g)(1)$$
$$= f''(x)+g''(x)+f'(x)+g'(x)+f(1)+g(1)$$
$$= f''(x)+f'(x)+f(1)+g''(x)+g'(x)+g(1)$$
$$= T(f(x)) + T(g(x))$$

Let $f \in P_3(\mathbb{R})$ and $c \in \mathbb{R}$ then

$$T(cf(x)) = (cf)''(x)+(cf)'(x)+cf(1)$$
$$= cf''(x)+cf'(x)+cf(1)$$
$$= c(f''(x)+f'(x)+f(1))$$
$$= cT(f(x)) \blacksquare$$

(b) (7 points)

$T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$

Let $\beta_2 = \{1, x, x^2\}$ $\beta = \{1, x, x^2, x^3\}$ be a basis for $P_2(\mathbb{R})$

$$T(1) = 1 = \begin{bmatrix}1\\0\\0\end{bmatrix}_{\beta_2}$$

$$T(x) = 0 - 2(1) + 1 = -1 = \begin{bmatrix}-1\\0\\0\end{bmatrix}_{\beta_2}$$

$$T(x^2) = 2 - 2(2x) + 1 = 3-4x = \begin{bmatrix}3\\-4\\0\end{bmatrix}_{\beta_2}$$

$$T(x^3) = 6x - 2(3x^2) + 1 = 1+6x-6x^2 = \begin{bmatrix}1\\6\\-6\end{bmatrix}$$ so

1	x	x^2	x^3
0	1	$2x$	$3x^2$
0	0	2	$6x$
0	0	0	6

Thus

$$[T]_\beta^{\beta_2} = \begin{bmatrix}1&-1&3&1\\0&0&-4&6\\0&0&0&-6\end{bmatrix}_\beta^{\beta_2}$$

To find $Q = [I]_{\beta_2}^{\gamma}$ we need to invert

$$\begin{bmatrix}1&1&0\\1&-1&0\\0&0&1\end{bmatrix}_\gamma^{\beta_2}$$

$$\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\1&-1&0&0&1&0\\0&0&1&0&0&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&-2&0&-1&1&0\\0&0&1&0&0&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&0&\frac{1}{2}&-\frac{1}{2}&0\\0&0&1&0&0&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&\frac{1}{2}&0\\0&1&0&\frac{1}{2}&-\frac{1}{2}&0\\0&0&1&0&0&1\end{array}\right]$$

$$Q^{-1} = \frac{1}{2}\begin{bmatrix}1&1&0\\1&-1&0\\0&0&2\end{bmatrix}$$

$$[T]_\beta^{\gamma} = \frac{1}{2}\begin{bmatrix}1&1&0\\1&-1&0\\0&0&2\end{bmatrix}\begin{bmatrix}1&-1&3&1\\0&0&4&6\\0&0&0&-6\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1&-1&7&7\\1&-1&-1&-5\\0&0&[illegible]&-12\end{bmatrix}$$

5. (10 points)

Let $[I]_\gamma^\beta = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

$[I]_\beta^\gamma = \left([I]_\gamma^\beta\right)^{-1}$

so

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -3 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{array}\right]$$

$$[I]_\beta^\gamma = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

So $[T]_\beta^\beta = [I]_\gamma^\beta \, [T]_\gamma^\gamma \, [I]_\beta^\gamma$

$$= \frac{1}{6}\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \boxed{\frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix} = [T]_\beta^\beta}$$

6. (a) (5 points)

If $ST = TS$ and v is an eigenvector of T w/ eigenvalue λ

Then $ST(v) = S(T(v)) = S(\lambda v)$

$= \lambda S(v)$ since S is linear

so since $ST = TS$

$ST(v) = TS(v) = \lambda S(v)$

so $T(S(v)) = \lambda S(v)$

Thus $S(v)$ is an eigenvector of T w/ eigenvalue λ. ∎

(b) (5 points)

We shall prove the following statement through its contrapositive, of if v is not an eigenvector of S, then λ does not have an algebraic multiplicity of 1.

if v is not an eigenvector, then

$S(v) = w \in V$ where $\lambda_1 v \neq w$ for any $\lambda_1 \in \mathbb{F}$

Since through part a), we can see for eigenvalue λ,

$T(v) = \lambda v$ and $T(S(v)) = \lambda S(v)$

3 the $\dim E_\lambda \geq 2$ and from a proposition proved in class we know that the algebraic multiplicity of $\lambda \geq \dim E_\lambda$

Thus alg mult of $\lambda \neq 1$ and we have proven the contrapositive ∎

7. (10 points)

$$A = \begin{pmatrix} 1 & -6 & 4 \\ -2 & -4 & 5 \\ -2 & -6 & 7 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -6 & 4 \\ -2 & -4-\lambda & 5 \\ -2 & -6 & 7-\lambda \end{pmatrix}$$

$$= (1-\lambda)((-4-\lambda)(7-\lambda)+30)$$
$$+2(-6(7-\lambda)+24)$$
$$-2(-6(5)-4(4-\lambda))$$
$$= (1-\lambda)(\lambda^2 \mp 3\lambda + 2)$$
$$+2(6\lambda - 18) - 2(4\lambda - 14)$$
$$= (-\lambda^3 + 3\lambda^2 - 2\lambda + \lambda^2 - 3\lambda + 2)$$
$$+ (4\lambda - 8)$$
$$= -\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

using rational zeroes, we can assume possible roots are 1, 2, 3

so using long division

$$\begin{array}{r} -\lambda^2 + \lambda + 2 \\ \lambda - 3 \overline{\smash{)}\, -\lambda^3 + 4\lambda^2 - \lambda - 6} \\ \underline{-\lambda^3 + 3\lambda^2} \\ \lambda^2 - \lambda \\ \underline{\lambda^2 - 3\lambda} \\ 2\lambda - 6 \end{array}$$

$$= (\lambda - 3)(-\lambda^2 + \lambda + 2)$$
$$= -(\lambda-3)(\lambda+1)(\lambda-2)$$

eigenvalues are 3, 2, -1

$\lambda = 3$

$$N(A - 3I) = N\left(\begin{pmatrix} 2 & -6 & 4 \\ -2 & -7 & 5 \\ -2 & -6 & 4 \end{pmatrix}\right) \rightsquigarrow N\begin{pmatrix} -2 & -6 & 4 \\ -2 & -7 & 5 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow N\begin{pmatrix} 1 & 3 & -2 \\ 2 & -7 & 5 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so for $\lambda = 3$ eigenvector is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda = 2$

$$N(A - 2I) = N\left(\begin{pmatrix} -1 & -6 & 4 \\ -2 & -6 & 5 \\ -2 & -6 & 5 \end{pmatrix}\right) \rightsquigarrow \begin{pmatrix} -1 & -6 & 4 \\ -2 & -6 & 5 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 6 & -4 \\ -2 & -6 & 5 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 6 & -4 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

so for $\lambda = 2$ eigenvector is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

It is diagonalizable in basis

$$\beta = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

w/ eigenvalues

3, 2, -1

so for $\lambda = -1$ eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = -1$

$$N(A - I) = \begin{pmatrix} 2 & -6 & 4 \\ -2 & -3 & 5 \\ -2 & -6 & 8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & -9 & 9 \\ 0 & -12 & 12 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} =$$

$\langle y,x\rangle = \overline{0} = 0$

$p \to q \equiv \sim q \to \sim p$

8. (10 points)

$\Rightarrow$ Given $\langle x,y\rangle = 0$

we must prove

$$\|x+cy\| \ge \|x\|$$

$$\Leftrightarrow \|x+cy\|^2 \ge \|x\|^2$$

$$\Leftrightarrow \langle x+cy, x+cy\rangle \ge \|x\|^2$$

$$\Leftrightarrow \langle x, x+cy\rangle + \langle cy, x+cy\rangle \ge \|x\|^2$$

$$\Leftrightarrow \overline{\langle x+cy, x\rangle} + c\overline{\langle x+cy, y\rangle} \ge \|x\|^2$$

$$\Leftrightarrow \overline{\langle x,x\rangle} + \overline{\langle cy,x\rangle} + c\overline{\langle x,y\rangle} + c\overline{\langle cy,y\rangle} \ge \|x\|^2$$

$$\Leftrightarrow \langle x,x\rangle + \bar{c}\langle x,y\rangle + c\langle y,x\rangle + c\bar{c}\langle y,y\rangle \ge \|x\|^2$$

Since $\langle x,y\rangle = 0$ and $\langle y,x\rangle = 0$

$$\Leftrightarrow \langle x,x\rangle + |c|^2\langle y,y\rangle \ge \|x\|^2$$

$$\|x\|^2 + |c|^2\|y\|^2 \ge \|x\|^2$$

~~Stator~~

$$|c|^2\|y\|^2 \ge 0$$

this will always be true

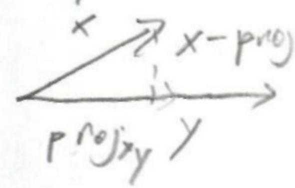
thus $\|x\| \le \|x+cy\|$ is always true if $\langle x,y\rangle = 0$

$\Leftarrow$ we shall prove this through the contrapositive in that if $\langle x,y\rangle \neq 0$, then there exists a $c \in F$ s.t. $\|x+cy\| < \|x\|$.

Since $x, y \in V$, we can write

$$x = \frac{\langle x,y\rangle}{\|y\|^2}\cdot y + x - \frac{\langle x,y\rangle}{\|y\|^2}y$$

or x a a projection on y and its orthogonal complement



Since $\langle x,y\rangle \neq 0$,

$$\frac{\langle x,y\rangle}{\|y\|^2}\cdot y \neq 0$$

and

$$x - \frac{\langle x,y\rangle}{\|y\|^2}y \neq x$$

Using pythagorean theorem

$$\left\|x - \frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 + \left\|\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 = \|x\|^2$$

so since $\left\|\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 > 0$

$$\left\|x - \frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 < \|x\|^2$$

So

$$\|x\| > \left\|x - \frac{\langle x,y\rangle}{\|y\|^2}y\right\|$$

thus if we set $c = -\frac{\langle x,y\rangle}{\|y\|^2}$, $\|x\| > \|x+cy\|$

and it existence proves the contraposite

9. (10 points)

Using the Gram-Schmidt process,

$V_1 = 1$ where $\|V_1\|^2 = 1\cdot 1 + 1\cdot 1 + 1\cdot 1$

$= 3$

$$V_2 = x - \frac{\langle x, 1\rangle}{3} 1 = x - \frac{1\cdot 1 + 2\cdot 1 + 3\cdot 1}{3} 1 = x - 2$$

$$\|V_2\|^2 = (x-2)(x-2) + (x-2)(x-2) + (x-2)(x-2)$$

$$= (1-2)^2 + (2-2)^2 + (3-2)^2$$

$$1 + 0 + 1$$

$$= 2$$

$$V_3 = x^2 - \frac{\langle x^2, 1\rangle}{3} 1 - \frac{\langle x^2, x-2\rangle}{2} (x-2)$$

$$= x^2 - \frac{1\cdot 1 + 1\cdot 4 + 1\cdot 9}{3} 1 - \frac{1\cdot -1 + 0\cdot 4 + 1\cdot 9}{2} (x-2)$$

$$= x^2 - \frac{14}{3} - 4x + 8$$

so

$$\beta = \left\{ 1, x-2, x^2 - 4x + 8 - \frac{14}{3} \right\}$$

is an orthogonal basis for V

10. (10 points)

Given the current inner product,
we need to find a $z \in P_2(\mathbb{R})$
where $\langle z, 1+x\rangle = \langle z, x+x^2\rangle = 0$

So
$W^\perp = \text{span}\{-5x^2+2x+1\}$

$z = ax^2+bx+c$

so

$$\langle ax^2+bx+c, 1+x\rangle = \int_{-1}^{1} ax^2+bx+c+ax^3+bx^2+cx \, dx$$
$$= \left[\frac{ax^3}{3}+\frac{bx^2}{2}+cx+\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}\right]_{-1}^{1}$$
$$= \frac{a}{3}+\frac{b}{2}+c+\frac{a}{4}+\frac{b}{3}+\frac{c}{2}-\left(-\frac{a}{3}+\frac{b}{2}-c+\frac{a}{4}-\frac{b}{3}+\frac{c}{2}\right)$$
$$= \frac{2a}{3}+\frac{2b}{3}+2c = 0$$

$$\langle ax^2+bx+c, x+x^2\rangle = \int_{-1}^{1} ax^3+bx^2+cx+ax^4+bx^3+cx^2 \, dx$$
$$= \left[\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+\frac{ax^5}{5}+\frac{bx^4}{4}+\frac{cx^3}{3}\right]_{-1}^{1}$$

2

$$= \frac{a}{4}+\frac{b}{3}+\frac{c}{2}+\frac{a}{5}+\frac{b}{4}+\frac{c}{3}-\left(\frac{a}{4}-\frac{b}{3}+\frac{c}{2}-\frac{a}{5}+\frac{b}{4}-\frac{c}{3}\right)$$
$$= \frac{2a}{5}+\frac{2b}{3}+\frac{2c}{3} = 0$$

so $\frac{2b}{3} = -\frac{2a}{3} - 2c$

$\frac{2a}{5} - \frac{2a}{3} - 2c + \frac{2c}{3} = 0$

$\frac{6}{15}a - \frac{10}{15}a = \frac{6}{3}c - \frac{2}{3}c$

$-\frac{4}{15}a = \frac{4}{3}c$

$-a = 5c$

$\boxed{a = -5c}$ $\boxed{c = \text{arbitrary}}$

So $\frac{2b}{3} = \frac{10c}{3} - 2c$

$\frac{2b}{3} = \frac{4}{3}c$ $\boxed{b = 2c}$

if x

11. (a) (4 points)

Let $x \in N(T)$ then $T(x)=0$ and $\|T(x)\|=0$ so $\|T(x)\|^2=0$

$= \langle T(x), T(x)\rangle$

$= \langle x, T^*(T(x))\rangle = 0$

If $x=0$, $T(x)=0$ and $T^*T(x)=0$

if $x \neq 0$, $\langle x, T^*(T(x))\rangle = 0$

$T^*(T(x)) = 0$ so $x \in N(T^*T)$

since

$N(T) \subseteq N(T^*T)$

$N(T^*T) \subseteq N(T)$

$N(T) = N(T^*T)$

Let $x \in N(T^*T)$ then $\langle x, T^*T(x)\rangle = 0$ and $\langle T(x), T(x)\rangle = 0$ so $\|T(x)\|^2 = 0$

$\Rightarrow \|T(x)\| = 0$

$\Rightarrow T(x) = 0$

$x \in N(T)$

(b) (2 points)

Using rank-nullity theorem and part 11a),

$\dim V = \dim R(T) + \dim N(T) \rightarrow$ $\dim R(T) + \dim N(T)$

$\dim V = \dim R(T^*T) + \dim N(T^*T) \rightarrow$ $= \dim R(T^*T) + \dim N(T^*T)$

since $\dim N(T^*T) = \dim N(T)$, we can cancel them out and result with

$\dim R(T) = \dim R(T^*T) \Rightarrow$

$\text{rank}(T) = \text{rank}(T^*T)$

(c) (4 points)

Using theorem proved in class, we know $W + W^\perp = V$ if W is a subspace of V

we shall first prove $R(T^*)^\perp = N(T)$

if $x \in N(T)$ and $y \in V$

then

$\langle T(x), y\rangle = 0 = \langle x, T^*(y)\rangle$

since $T^*(y) \in R(T^*)$

$x \in R(T^*)^\perp$

if $x \in R(T^*)^\perp$,

$\|T(x)\|^2 = \langle T(x), T(x)\rangle$

$= \langle x, T^*(T(x))\rangle$

since $T^*(T(x)) \in R(T^*)$

$= 0$

Thus

$\|T(x)\| = 0$

and

$T(x) = 0$

so $x \in N(T)$

Thus we have proven

$R(T^*)^\perp = N(T)$

so

$\dim R(T^*) + \dim R(T^*)^\perp = V$

and

$\dim R(T) + \dim N(T) = V$

$\dim R(T^*) + \dim R(T^*)^\perp$

$= \dim R(T) + \dim N(T)$

canceling $R(T^*)^\perp$ and $N(T)$

$\dim R(T^*) = \dim R(T) \Rightarrow$

$\text{rank}\, T^* = \text{rank}\, T$

12. (a) (5 points)

Let λ be an arbitrary eigenvalue with associated eigenvector v

if $K=1$

$T(v) = \lambda v = 0$ since $v \neq 0$, $\lambda = 0$

if $K>1$,

since T is linear

$$0 = T^K(v) = \underbrace{T \circ T \circ \cdots \circ T}_{K}(v) = \underbrace{T \circ \cdots \circ T}_{K-1}(\lambda v) = \lambda \underbrace{T \circ \cdots \circ T}_{K-1}(v) = \lambda \underbrace{T \circ \cdots \circ T}_{K-2}(\lambda v)$$

$$= \lambda^2 \underbrace{T \circ \cdots \circ T}_{K-2}(v) = \cdots = \lambda^K v = 0$$

continue as K reaches 0

Since $v \neq 0$, $\lambda^K = 0$

Since all arbitrary $\lambda = 0$,

0 is the only eigenvalue

(b) (5 points)

Since V is a finite dimensional complex inner product space and T is normal, from a theorem proved in class, we know there exists an orthonormal basis of V which with eigenvectors of T. we shall call β

Thus we can diagonalize T with this basis β so that T is just the eigenvalues of T.

Since $T^K = T_0$, we know that 0 is the only eigenvalue, so

$[T]_\beta = [0]_\beta$ which means any $[v]_\beta \in V$ will be sent to 0

So $T = T_0$

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