Directions. Write or type the answers to the following problems neatly on blank paper. Most problems will require some scratch work, but you should only write your final answers on your submission. Do not turn in your scratch work. **In the work you do submit, you need to justify everything you write, and everything you write must be in full sentences.**

Due date: This exam is due at 12:00am Pacific Time on Saturday Novemeber 21st (so one minute past **11:59pm on Friday November 20th**). Note Gradescope is programmed to stop accepting submissions at 12:00am and will not accept anything after 12:00am. I recommend leaving at least 20 minutes to upload the exam just in case something goes wrong.

Materials allowed: You are allowed to use any course materials (notes, the textbook, anything on the CCLE page). You are also allowed to use a calculator to help with computations, but you should not be looking up answers in other books or online. You also must work on the exam alone. You cannot receive assistance from any peers, TAs, instructors, people on the internet, or anyone else.

How to contact us if you have issues: Campuswire is the best place to ask questions, but please **do not make any public posts on Campuswire during the exam**. To get the quickest response, you should make a private post that can only be seen by the instructor and the TAs. This way one of the four of us can answer. You can also direct message your TA or the instructor. Lastly, you can send an email to your TA or to the instructor.

Note if you ask a question after 7pm on Friday evening, then it is possible your question will not get answered until the following day.

Honesty Statement.

(***) Write the following statement at the top of your exam:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

and then sign your name. **This statement must be written on your exam with your signature. If the statement or your signature is missing, then your exam will not be graded.**

Exam Problems. There are 35 points available on the exam. **You must write in full sentences and justify everything you write, even for the more computational problems.** You cannot cite any previous homework problems or quiz problems, but you are allowed to cite any theorems, lemmas, corollaries, or properties given in the textbook sections through Section 5.1, even if the proof is left as an exercise.

1. Prove or disprove the following statements.

   (a) [3 pts] Let V be the subspace of $P_2(\mathbb{R})$ given by $V = \{a + bx + cx^2 \mid a + b + c = 0\}$. There exists an isomorphism $T : V \to \mathbb{R}^3$.

   (b) [3 pts] Every linear map $S : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}^4$ is an isomorphism.

2. Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be the linear map defined by $T(f(x)) = f(2)x + 3f'(x)$. Let $p(x) = 4 + 6x$ and $\gamma = \{1, x\}$.

   (a) [3 pts] Compute $T(p(x))$ using the definition of T. Then find $[T(p(x))]_\gamma$.

   (b) [3 pts] Find $[T]_\gamma^\gamma$.

   (c) [4 pts] Compute $[p(x)]_\gamma$ and then use matrix multiplication to compute $[T]_\gamma^\gamma [p(x)]_\gamma$. Verify that it equals $[T(p(x))]_\gamma$.

3. Let T be the linear operator defined in Problem 2.

   (a) [3 pts] Find all eigenvalues of T. You can use what you computed in Problem 2.

   (b) [5 pts] Find a basis β of eigenvectors for T. What is $[T]_\beta^\beta$?

   (c) [3 pts] Is there a basis α for $P_1(\mathbb{R})$ such that $[T]_\alpha^\alpha = \begin{pmatrix} 12 & 4 \\ 0 & 5 \end{pmatrix}$? Justify your answer.

4. Suppose V be a vector space over a field $\mathbb{F}$ with $\dim(V) = n$. Let α, β, and γ be ordered bases for V. Let P denote the matrix that changes α coordinates into β coordinates, and let Q denote the matrix that changes β coordinates to γ coordinates.

   (a) [4 pts] Prove that Q^{-1} is the matrix that changes γ coordinates into β coordinates.

   (b) [4 pts] Recall the linear functions $\phi_\alpha : V \to \mathbb{F}^n$ and $\phi_\gamma : V \to \mathbb{F}^n$ are given by $\phi_\alpha(x) = [x]_\alpha$ and $\phi_\gamma(x) = [x]_\gamma$. Prove that $L_Q L_P \phi_\alpha = \phi_\gamma$.