Directions. Write the answers to the following problems on clean paper. Most problems will require some scratch work, so you should have some scratch paper next to you during the exam. Do not turn in your scratch work. **In the work you do submit, you need to justify everything you write, and everything you write must be in full sentences.**

Due date: This exam is due at 12:00am Pacific Time on Saturday October 31st (so one minute past **11:59pm on Friday October 30th**). Note Gradescope is programmed to stop accepting submissions at 12:00am and will not accept anything after 12:00am. I recommend leaving at least 20 minutes to upload the exam just in case something goes wrong.

Materials allowed: You are allowed to use any course materials (notes, the textbook, anything on the CCLE page). You should not be looking up answers in other books or online. You also must work on the exam alone. You cannot receive assistance from any peers, TAs, instructors, people on the internet, or anyone else.

How to contact us if you have issues: Campuswire is the best place to ask questions, but please **do not make any public posts on Campuswire during the exam**. To get the quickest response, you should make a private post that can only be seen by the instructor and the TAs. This way one of the four of us can answer. You can also direct message your TA or the instructor. Lastly, you can send an email to your TA or to the instructor.

Note if you ask a question after 7pm on Friday evening, then it is possible your question will not get answered until the following day.

Honesty Statement.

(***) Write the following statement at the top of your exam:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

and then sign your name. **This statement must be written on your exam with your signature. If the statement or your signature is missing, then your exam will not be graded.**

Exam Problems. There are 35 points available on the exam.

1. [6 pts] Prove or disprove the following statements.
   (a) The subset $U = \{(a_1, a_2) \in \mathbb{R}^2 \ : \ a_1 = -4a_2\}$ is a subspace of $\mathbb{R}^2$.
   (b) The subset $W = \left\{ \begin{pmatrix} i & a \\ b & 0 \end{pmatrix} \ : \ a, b \in \mathbb{C} \right\}$ is a subspace of $M_{2\times 2}(\mathbb{C})$.
2. [5 pts] Suppose V is a vector space over a field $\mathbb{F}$ and let u, v be distinct vectors in V and a be a nonzero scalar in $\mathbb{F}$. Prove that if $\{u, v\}$ is a basis for V, then $\{u, u + av\}$ is a basis for V.
3. [5 pts] Is there a linear map $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ such that $T(1) = (2, 3)$, $T(1+x) = (-2, 7)$, and $T(1+x+x^2) = (0, 9)$? Justify your answer.

4. Suppose U, V, and W are finite dimensional vector spaces over a field $\mathbb{F}$. Let $T : U \to V$ and $S : V \to W$ be linear transformations, and suppose $S \circ T = T_0$ where T_0 is the zero transformation, that is $T_0(u) = 0_W$ for all $u \in U$.

   (a) [3 pts] Prove $R(T) \subseteq N(S)$.

   (b) [6 pts] Prove that if T is injective and S is surjective, then $\dim(W) + \dim(U) \leq \dim(V)$. (Hint: try to use the Dimension Theorem and part (a) in your proof.)

5. For each of the following subspaces, write down a basis and then state the dimension. For this problem only, you do not have to provide justification for your answer. That is, you do not have to prove your sets are bases.

   (a) [2 pts] The subspace V of $P_3(\mathbb{R})$ defined by $V = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{R}) \ : \ a_0 + a_1 + a_2 + a_3 = 0\}$.

   (b) [2 pts] The subspace W of $M_{3\times 3}(\mathbb{C})$ consisting of all matrices A such that the diagonal entries in A are zero. That is,
   $$W = \{A \in M_{3\times 3}(\mathbb{C}) \ : \ A_{ij} = 0 \text{ whenever } i = j\}.$$
   (The subspace W is being considered as a vector space over the field $\mathbb{C}$ in this example.)

6. Let V be a vector space over a field $\mathbb{F}$.

   (a) [3 pts] The statement below is false. Disprove the statement below by providing a counterexample. Make sure to explain why your example is a counterexample.

   If S is a linearly dependent subset of V, then every vector in S can be written as a linear combination of other vectors in S.

   (b) [3 pts] Your classmate thinks this statement is true and that they wrote down a proof. Your counterexample shows it's false, but they can't find any errors in their proof below. Find the error and then write a sentence or two explaining why it is an error.

   "Proof". Suppose S is a linearly dependent set. Then by definition of linearly dependent, there exist distinct vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n \in \mathbb{F}$ that are not all zero such that
   $$a_1u_1 + \cdots + a_nu_n = 0.$$
   Since the scalars are not all zero, there exists some index i such that $a_i \neq 0$. Thus we can multiply both sides by a_i^{-1} and subtract to get
   $$u_i = -a_i^{-1}a_1u_1 - \cdots - a_i^{-1}a_{i-1}u_{i-1} - a_i^{-1}a_{i+1}u_{i+1} - \cdots - a_i^{-1}a_nu_n.$$
   Thus u_i can be written as a linear combination of other elements in S. Since u_i is an arbitrary element in S, we can conclude every element in S can be written as a combination of other elements in S. $\square$