MATH 115A Midterm II, Fall 2018

Name:

Justify All Your Answers.

Problem 1. (5)

Let T be the linear transformation from $\mathbf{P}_2(\mathbf{R})$ to itself defined by $T(f) = 3f + 2f' + f''$, and $\alpha = \{v_1, v_2, v_3\}$ be an ordered basis. Here $v_1 = 1 + x + x^2$, $v_2 = x - x^2$ and $v_3 = x^2$.

(i)Find $[T]_\beta$, where $\beta = \{1, x, x^2\}$ is the standard basis of $\mathbf{P}_2(\mathbf{R})$.

(ii) Find the matrix $[T]_\alpha$ by using the formula relating $[T]_\beta$ and $[T]_\alpha$.

Note: For (ii), no points will be given without using the formula.

(i) $T(\beta) = (T(1), T(x), T(x^2)) = (3, 3x+2, 3x^2+4x+2)$

$$= (1, x, x^2) \underbrace{\begin{bmatrix} 3 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}}_{[T]_\beta}$$

2 pts

(ii) $$\alpha = (1+x+x^2, x-x^2, x^2) = (\underbrace{1, x, x^2}_{\beta}) \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{Q}$$

$$Q^{-1}: \quad [I : Q] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

1pt

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ \underbrace{-2 \;\; 1 \;\; 1}_{Q^{-1}} & & & 0 & 0 & 1 \end{array}\right]$$

$$[T]_\alpha = Q^{-1} T_\beta Q = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

2 pts

$$= \begin{bmatrix} 3 & 2 & 2 \\ -3 & 1 & 2 \\ -6 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 2 \\ 0 & -1 & 2 \\ -4 & -4 & 3 \end{bmatrix}$$

Problem 2. (5)

Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

(i) Find eigenvalues of A and a basis for each eigenspace over $\mathbf{R}$ and $\mathbf{C}$ respectively.

(ii) Decide if A is diagonalizable over $\mathbf{R}$ and $\mathbf{C}$ respectively. If it is, find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$.

(iii) Find A^n.

(i) $f_A(\lambda) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 - 4\lambda + 5 = 0$

$\Longrightarrow \lambda_\pm = \dfrac{4 \pm \sqrt{16-20}}{2} = \boxed{2 \pm i}$

$\Longrightarrow$ • E_{λ_+}: $0 = (A - \lambda_+ I)X = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $E_{\lambda_+} = \operatorname{span}\left\{ \overset{v_+}{\begin{bmatrix} 1 \\ -i \end{bmatrix}} \right\}$

3 pts • $E_{\lambda_-} = \operatorname{span}\left\{ \overset{v_-}{\begin{bmatrix} 1 \\ i \end{bmatrix}} \right\}$ over $\mathbb{C}$.

• There is no real eigenvalue.

(ii) A is diagonalizable over $\mathbb{C}$ but not digonalizable over $\mathbb{R}$.

1 pt

Over $\mathbb{C}$, $A = SDS^{-1} = \underbrace{\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}}_{S} \begin{bmatrix} 2+i & \\ & 2-i \end{bmatrix} \underbrace{\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} / 2i}_{S^{-1}}$

(iii) $\therefore$ $A^n = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} (2+i)^n & \\ & (2-i)^n \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$

1 pt

(ii) $\Rightarrow E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$\circ\ \lambda_2 = 3$, $\text{mult}(\lambda_2) = 1$, $E_{\lambda_2}: \ 0 = (A - \lambda_2 I)X = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

2 pts $\tilde{E}_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ & $E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.

(iii) T is diagonalizable since $\sum_i \cancel{\text{geom}}\,\text{mult}(\lambda_i) = 3$.

1 pt

Problem 3. (5)

Let V be the vector space of 2 by 2 upper triangular matrices with entries in $\mathbf{R}$, and

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}.$$

Define the linear transformation T form V to itself by the formula $T(X) = XA$ for any X in V.

(i) Find eigenvalues of T and their multiplicities.

(ii) Find a basis for each eigenspace of T and the dimension of each eigenspace.

(iii) Decide if T is diagonalizable.

std basis for V, $\beta = \{e_{11}, e_{12}, e_{22}\}$

2 pts

$$T(\beta) = (T(e_{11}), T(e_{12}), T(e_{22}))$$

$$= (e_{11} \cdot A, e_{12} \cdot A, e_{22} \cdot A)$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \underbrace{(e_{11}, e_{12}, e_{22})}_{\beta} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[T]_\beta = A}$$

(i) $f_T(\lambda) = f_A(\lambda) = (1-\lambda)(\lambda^2 - 4\lambda + 3)$
$= (1-\lambda)^2(\lambda - 3)$

(ii) $\circ\ \lambda_1 = 1$, $\text{mult}(\lambda_1) = 2$, $E_{\lambda_1}: \ 0 = (A - \lambda_1 I)X$

$= \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \tilde{E}_{\lambda_1} = \text{span}\left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\tilde{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\tilde{v}_2} \right\}$

$\Longrightarrow K = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ & $\langle L, X \rangle = l_{11}a + l_{12}b + l_{21}c + l_{22}d = 3a + 4b + 5c + 6d$

$\Longrightarrow L = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ $\therefore$ $A_1 = \frac{K+L}{2} = \boxed{\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}}$

$A_2 = \frac{K-L}{2} = \boxed{\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}}$

Problem 4. (5)

Let $V = M_{2\times 2}(\mathbf{R})$ be the vector space of 2 by 2 matrices with the inner product defined by $< A, B >= tr(AB^t)$ for any A and B in V, where B^t is the transpose of B. Let $T : V \to \mathbf{R}^2$ be the linear tarnsformatin defined as follows. For

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $V = M_{2\times 2}(\mathbf{R})$, $T(X) = (a + 2b + 3c + 4d, 3a + 4b + 5c + 6d) \in \mathbf{R}^2$.

(i) Show that there are unique elements A_1 and A_2 in V such that for any X in V the following identity holds: $T(X) = (< A_1 + A_2, X >, < A_1 - A_2, X >)$.

(ii) Find the A_1 and A_2 in (i).

Uniqueness:

If $(\langle A_1 + A_2, X \rangle, \langle A_1 - A_2, X \rangle)$

$= (\langle A_1' + A_2', X \rangle, \langle A_1' - A_2', X \rangle)$ for $\forall X$

$\Longrightarrow$ Let $B_1 = A_1 - A_1'$ & $B_2 = A_2 - A_2'$ then

2 pts

$(\langle B_1 + B_2, X \rangle, \langle B_1 - B_2, X \rangle) = \langle 0, 0 \rangle$

$\Longrightarrow 2\langle B_1, X \rangle = \langle B_1 + B_2, X \rangle + \langle B_1 - B_2, X \rangle$

$= 0$

& $2\langle B_2, X \rangle = \langle B_1 + B_2, X \rangle - \langle B_1 - B_2, X \rangle = 0$

for $\forall X \Longrightarrow \langle B_1, B_1 \rangle = 0$; $\langle B_2, B_2 \rangle = 0$

hence $B_1 = A_1 - A_1' = 0$ $B_2 = A_2 - A_2' = 0$

Existence: Let $A_1 + A_2 = K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ &

3 pts

$A_1 - A_2 = L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$

Then $\langle K, X \rangle = k_{11}a + k_{12}b + k_{21}c + k_{22}d = a + 2b + 3c + 4d$