20S-MATH115A-4 Submission Midterm

ANDREW NG

TOTAL POINTS

100 / 100

QUESTION 1

1 Problem 1 25 / 25

✓ **+ 25 pts** Correct

- **+ 7 pts** Found RREF of the matrix
- **+ 7 pts** Found correct basis for N(T)
- **+ 4 pts** Found partially correct basis for N(T)
- **+ 7 pts** Found correct basis for R(T)
- **+ 4 pts** Found partially correct basis for R(T)
- **+ 4 pts** Found rank and nullity
- **- 1 pts** Typo or minor mistakes

QUESTION 2

2 Problem 2 25 / 25

- **+ 6 pts** (a) correct
- **+ 6 pts** (b) correct
- **+ 6 pts** (c) correct
- **+ 7 pts** (d) correct
- **+ 3 pts** (a) partially correct
- **+ 3 pts** (b) partially correct
- **+ 3 pts** (c) partially correct
- **+ 4 pts** (d) partially correct

✓ **+ 25 pts** Correct

QUESTION 3

3 Problem 3 25 / 25

✓ **- 0 pts** Correct

- **- 3 pts** Typo or calculation errors
- **- 15 pts** Wrong matrix representation

QUESTION 4

4 Problem 4 25 / 25

✓ **+ 25 pts** Correct

- **+ 4 pts** N(T)
- **+ 4 pts** R(T)
- **+ 4 pts** Basis for N(T)
- **+ 4 pts** Basis for R(T)
- **+ 1 pts** Rank and Nullity
- **+ 4 pts** Intersection of N(T) and R(T)
- **+ 4 pts** N(T) + R(T)
- **- 1 pts** Minor mistakes

Problem 1. (25)

Let T be a linear transformation from $\mathbf{R}^4$ to itself defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 3 \\ 3 & 4 & 7 & 9 \\ 4 & 5 & 9 & 12 \end{bmatrix}$$

and $\mathbf{x}$ is a column vector in $\mathbf{R}^4$.
(i) Find a basis of the null space of T.

(ii) Find a basis of the range of T.

(iii) What are the nullity and rank of T?

(i)

$$(A|0) = \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 3 & 0 \\ 3 & 4 & 7 & 9 & 0 \\ 4 & 5 & 9 & 12 & 0 \end{array}\right) \xrightarrow[(\mathrm{IV}-4\mathrm{I})]{\substack{(\mathrm{I}) \\ (\mathrm{II}-\mathrm{I}) \\ (\mathrm{III}-3\mathrm{I})}} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \Rightarrow \overbrace{\left(\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)}^{rref(A)}$$

$$\forall \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in N(T), \quad \begin{aligned} x_1 + x_3 + 3x_4 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -s - 3t \\ x_2 &= -s \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

let x_3 be arbitrary s

let x_4 be arbitrary t

$$\Rightarrow \forall x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in N(T), \quad x = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} t \Rightarrow \boxed{\text{basis of nullspace of } T: \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

(ii) We know that $rref(A) = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

We know that a basis of $R(A)$ will be formed by the coloumns of the "pivotal" variables where there are leading 1's. (which are the first 2 here)

$$\Rightarrow \boxed{\text{basis of } R(T) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix} \right\}}$$

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(iii) $nullity(T) = \#(\text{basis of } N(T)) = 2$

$rank(T) = \#(\text{basis of } R(T)) = 2$

$$\Rightarrow \boxed{\begin{aligned} nullity(T) &= 2 \\ rank(T) &= 2 \end{aligned}}$$

Problem 2. (25)

Let V be a vector space over a field F. Suppose that W_1 and W_2 are two subspaces, neither of them is contained in the other. Prove or disprove the following statements:

(a) $W_1 \cup W_2$ is a subspace;

(b)$W_1 + W_2$ is a subspace, where $W_1 + W_2$ is defined to be the collection of elements of the form $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(c) any element z in $W_1 + W_2$ can be uniquely expressed as $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(d) $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$ if $W_1 \cap W_2 = \{0\}$.

(a) This is not necessarily true. Here is a counterexample:

let $V = \mathbb{R}^2$, $W_1 = \text{span}\{\begin{pmatrix}1\\0\end{pmatrix}\}$ and $W_2 = \text{span}\{\begin{pmatrix}0\\1\end{pmatrix}\}$, and $F = \mathbb{R}$

let $x = \begin{pmatrix}1\\0\end{pmatrix} \in W_1 \subseteq W_1 \cup W_2$

$y = \begin{pmatrix}0\\1\end{pmatrix} \in W_2 \subset W_1 \cup W_2$

$x + y = \begin{pmatrix}1\\1\end{pmatrix} \notin W_1 \cup W_2$, so $W_1 \cup W_2$ is not closed under addition so $W_1 \cup W_2$ is not necessarily a subspace

(b) Proof:

Obviously, $W_1 + W_2 \subseteq V$ because $\forall x \in W_1, y \in W_2$, $x + y \in V$ because V is closed under addition.

$0 \in W_1$, $0 \in W_2$ because W_1 and W_2 are subspaces, so $0 + 0 = 0 \in W_1 + W_2$; so the zero element is in $W_1 + W_2$

let $c, d \in F$, $z_1, z_2 \in W_1 + W_2$, $x_1, x_2 \in W_1$, $y_1, y_2 \in W_2$

where $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$

$cz_1 + dz_2 = c(x_1 + y_1) + d(x_2 + y_2) = cx_1 + cy_1 + dx_2 + dy_2$

$(cx_1 + dx_2) \in W_1$ and $(cy_1 + dy_2) \in W_2$ because W_1 and W_2 closed

$cz_1 + dz_2 = (cx_1 + dx_2) + (cy_1 + dy_2) \in W_1 + W_2 \Rightarrow W_1 + W_2$ is closed

Because $W_1 + W_2$ includes 0 of V and is closed under addition and scalar multiplication, $W_1 + W_2$ is a subspace.

(2) (c) This is not necessarily true. Here is a counterexample:

Let $V = \mathbb{R}^3$ over $\mathbb{R}$, $W_1 = \text{span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right)$, $W_2 = \text{span}\left(\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right)$.

let $z = \begin{pmatrix}1\\1\\1\end{pmatrix} \in W_1 + W_2$

$z = x_1 + y_1$ where $x_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \in W_1$ and $y_1 = \begin{pmatrix}0\\1\\1\end{pmatrix} \in W_2$

But, $z = x_2 + y_1$ where $x_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} \in W_1$ and $y_2 = \begin{pmatrix}0\\0\\1\end{pmatrix} \in W_2$

Because $x_1 \neq x_2$, $y_1 \neq y_2$, $z \in W_1 + W_2$ and is not uniquely expressed as $z = x + y$ with $x \in W_1$ and $y \in W_2$

(d) Proof:

We can use the fact that $\dim(W_1 + W_2) = \dim(W_1) + \dim W_2 - \dim(W_1 \cap W_2)$

$W_1 \cap W_2 = \{0\} \Rightarrow \dim(W_1 \cap W_2) = 0$

$\Rightarrow \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$

Problem 3. (25)

Given two matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, define two linear transformations T_A and T_B form $M_{2,2}(\mathbf{R})$ to itself by the formula $T_A(X) = AX$ and $T_B(X) = BX$ for $X \in M_{2,2}(\mathbf{R})$. Let $\beta = \{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$ be the standard basis of $M_{2,2}(\mathbf{R})$.

(i) Find the matrices $[T_A]_\beta$ and $[T_B]_\beta$.

(ii) Find the matrices $[T_A \circ T_B]_\beta$ and $[T_B \circ T_A]_\beta$.

(iii) Find the matrix $[(T_A \circ T_B - T_B \circ T_A)^n]_\beta$, where $(T_A \circ T_B - T_B \circ T_A)^n = (T_A \circ T_B - T_B \circ T_A) \circ (T_A \circ T_B - T_B \circ T_A) \cdots (T_A \circ T_B - T_B \circ T_A)$ is the n-fold composition of $(T_A \circ T_B - T_B \circ T_A)$.

(i) $T_A\beta = (T_A(e_{1,1}), T_A(e_{1,2}), T_A(e_{2,1}), T_A(e_{2,2})) = \left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$

$= \left(0, 0, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right) = (0, 0, e_{1,1}, e_{1,2})$

$\boxed{[T_A]_\beta = \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}}$

$T_B\beta = (T_B(e_{1,1}), T_B(e_{1,2}), T_B(e_{2,1}), T(e_{2,2})) = \left(\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$

$= \left(\begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 0, 0\right) = (e_{2,1}, e_{2,2}, 0, 0)$

$\boxed{[T_B]_\beta = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix}}$

(ii) $[T_A \circ T_B]_\beta = [T_A]_\beta \cdot [T_B]_\beta = \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix} = \boxed{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} = [T_A \circ T_B]_\beta}$

$[T_B \circ T_A]_\beta = [T_B]_\beta \cdot [T_A]_\beta = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix} = \boxed{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = [T_B \circ T_A]_\beta}$

Let $T_Q = T_A \circ T_B - T_B \circ T_A$

(iii) First, we can establish a few things:

$$[(T_A \circ T_B) \circ (T_A \circ T_B)]_\beta = [T_A \circ T_B]_\beta \cdot [T_A \circ T_B]_\beta = [T_A \circ T_B]_\beta$$

$$[(T_B \circ T_A) \circ (T_B \circ T_A)]_\beta = [T_B \circ T_A]_\beta \cdot [T_B \circ T_A]_\beta = [T_A \circ T_B]_\beta$$

$$[(T_B \circ T_A) \circ (T_A \circ T_B)]_\beta = [T_B \circ T_A]_\beta \cdot [T_A \circ T_B]_\beta = 0$$

$$[(T_A \circ T_B) \circ (T_B \circ T_A)]_\beta = [T_A \circ T_B]_\beta \cdot [T_B \circ T_A]_\beta = 0$$

For $n = 2$, $(T_Q)^2 = (T_A \circ T_B - T_B \circ T_A) \circ (T_A \circ T_B - T_B \circ T_A)$

$$= (T_A \circ T_B - T_B \circ T_A) \circ (T_A \circ T_B) - (T_A \circ T_B - T_B \circ T_A)(T_B \circ T_A)$$

$$= (T_A \circ T_B \circ T_A \circ T_B) - (T_B \circ T_A \, T_A \circ T_B) - (T_A \circ T_B \, T_B \circ T_A) + (T_B \circ T_A \circ T_B \circ T_A)$$

$$[(T_Q)^2]_\beta = [T_A T_B]_\beta - 0 - 0 + [T_B \circ T_A]_\beta$$

$$[(T_Q)^3]_\beta = ([T_A T_B]_\beta + [T_B T_A]_\beta) \cdot ([T_A T_B]_\beta - [T_B T_A]_\beta)$$

$$= \underbrace{[T_A T_B]_\beta \cdot [T_A T_B]_\beta}_{} - \underbrace{[T_A T_B]_\beta \cdot [T_B T_A]_\beta}_{} + \underbrace{[T_B T_A]_\beta \cdot [T_A T_B]_\beta}_{} - \underbrace{[T_B T_A]_\beta [T_B T_A]_\beta}_{}$$

$$= [T_A T_B]_\beta - 0 + 0 - [T_B T_A]_\beta$$

$$= [T_Q]_\beta$$

$\Rightarrow$ Incrementing n by 1 alternates $[T_Q^n]_\beta$ between $[T_Q]_\beta$ and $[T_A \circ T_B + T_B \circ T_A]_\beta$

when n is odd: $[T_Q^n]_\beta = [T_A \circ T_B - T_B \circ T_A]_\beta = \left[\begin{pmatrix} e_{1,1} \\ e_{1,2} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ e_{2,1} \\ e_{2,2} \end{pmatrix}\right]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

when n is even $[T_Q^n]_\beta = [T_A \circ T_B + T_B \circ T_A]_\beta = \left[\begin{pmatrix} e_{1,1} \\ e_{1,2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e_{2,1} \\ e_{2,2} \end{pmatrix}\right]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow \boxed{[(T_A \circ T_B - T_B \circ T_A)^n]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^n & 0 \\ 0 & 0 & 0 & (-1)^n \end{pmatrix}}$$

Problem 4. (25)

Let $V = M_{n,n}(\mathbf{R})$ be the vector space of n by n matrices with entries in the field $\mathbf{R}$ and $T : V \to V$ be a linear transformation defined by $T(X) = X - X^t$ for $X \in V$, where X^t is the transpose of X.

(i) Find the null space $N(T)$ and the range $R(T)$.

(ii) Find a basis for $N(T)$ and $R(T)$ respectively. What are the nullity and rank of T?

(iii) What are the intersection $N(T) \cap R(T)$ and the sum $N(T) + R(T)$?

(i) $(X - X^T)_{ij} = X_{ij} - X^T_{ij} = X_{ij} - X_{ji}$

$\boxed{N(T) = \{ v \in V : v_{ij} = v_{ji} \ \forall \ 1 \le i,j \le n \}}$ or matrices where symmetric matrices

If $i=j$, $(X - X^T)_{ii} = X_{ii} - X_{ii} = 0 \Rightarrow$ 0's along diagonal

$T(X)_{ij} = (X - X^T)_{ij} = X_{ij} - X_{ji} = -(X_{ji} - X_{ij}) = -(X - X^T)_{ji} = -T(X)_{ji} \Rightarrow$ skew symmetric

$\boxed{R(T) = \{ v \in V : v_{ij} = -v_{ji} \ \forall \ 1 \le i,j \le n \}}$, the space of all $n \times n$ skew symmetric matrices, or matrices where $X^T = -X$

(ii) basis of $N(T)$: $\{ E_{ii} : 1 \le i \le n \} \cup \left(\{ v \in V : v_{ij} = v_{ji} = 1 \} \ \forall \ 1 \le i,j \le n \} \right)$ ← contains duplicates

$\left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \\ \vdots & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}, \text{etc} \right\}$ $\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & \ddots & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & \ddots & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{etc} \right)$

can be combined to: because doesn't matter if some on diagonal have magnitude 2

$\boxed{\{ (E_{ij} + E_{ji}) \text{ for } 1 \le i \le n, \ i \le j \le n \} \text{ is basis of } N(T)}$

$1 \le i \le n$ $\quad$ $i \le j \le n \to n, n-1, \ldots, 1$ elements per i, which averages to

$\left(n \text{ poss values of } i \right) \times \left(\frac{n+1}{2} \right)$ values of j for each i

$$\#(\text{basis of } N(T)) = \boxed{\frac{n^2 + n}{2} = \text{nullity of } T}$$

4 ii continued

basis of $R(T)$: all elements of diagonal must be 0, and elements of basis should be matrices where $m \in V$, $m_{ij} = 1 = -m_{ji} = -1$ for $i \neq j$, $1 \leq i \leq n$, $j > i$ excludes elements on diagonal

$$\boxed{\text{basis of } R(T) : \{(E_{ij} - E_{ji}) \text{ for } 1 \leq i \leq n, i < j \leq n\}}$$

$1 \leq i \leq n$ $\quad$ $i < j \leq n \rightarrow n-1, n-2, \ldots, 1, 0$ elements per i, which averages to $\frac{n-1}{2}$

$\left(\begin{array}{c} n \text{ values} \\ \text{of } i \end{array}\right) \times \left(\frac{n-1}{2}\right)$

$\Rightarrow \boxed{\text{rank of } T = \frac{n^2-n}{2}} = \#(\text{basis of } R(T))$

(iii) $N(T) \cap R(T)$

$v_{ij} = 0$ for $1 \leq i \leq n$

$v_{ij} = v_{ji}$ for $1 \leq j, i \leq n$

$v_{ij} = -v_{ji}$ for $1 \leq j, i \leq n$

$v_{ij} = -v_{ij} \Rightarrow v_{ij} = 0$ for $1 \leq j, i \leq n \Rightarrow \boxed{N(T) \cap R(T) = \{0\}}$

$N(T) + R(T)$

All $E_{ij} \in N(T) + R(T)$ for $1 \leq i \leq n$, $1 \leq j \leq m$ because:

For $i = j$, $E_{ij} \in N(T)$ because $v_{ij} = v_{ii} = v_{ji}$ for such matrices

For $i \neq j$, $E_{ij} = (E_{ij} + E_{ji} + E_{ij} - E_{ji})/2$, and $E_{ij} + E_{ji} \in N(T)$, $E_{ij} - E_{ji} \in R(T)$, so E_{ij} can be written as linear combination of elements in $N(T)$ and $R(T)$, so it is in $N(T) + R(T)$

Because all elements of the standard basis of V are in $N(T) + R(T) \Rightarrow V \subseteq N(T) + R(T)$

Because $N(T) \subseteq V$ and $R(T) \subseteq V$, and V is a vector space so it is closed under addition, any $n \in N(T) \subseteq V + r \in R(T) \subseteq R$ is still in $V \Rightarrow N(T) + R(T) \subseteq V$

$$\boxed{\text{Thus, } N(T) + R(T) = V.}$$