Yifan You

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MATH 115A Midterm I, Spring 2019

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Justify All Your Answers

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Problem 1. (5)

Let T be a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 12 & 6 & 9 \\ 5 & 14 & 10 & 9 \end{bmatrix}$$

and $\mathbf{x}$ is a column vector in $\mathbf{R}^4$.

(i) Find a basis of the null space of T.

(ii) Find a basis of the range of T.

(iii) What is nullity and rank of T?

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rank-nullity

rank(A) is equal to # of leading 1s in rref

$\text{rank}(A) = \dim(R(A)) = 2$

$\dim(N(A)) = \cancel{4} - \text{rank}(A) = 4 - 2 = 2$

(i) $\begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 12 & 6 & 9 \\ 5 & 14 & 10 & 9 \end{bmatrix} \begin{matrix} \\ -3(\text{I}) \\ -5(\text{I}) \end{matrix}$

$\downarrow$

$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & -6 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \div -6$

$\downarrow$

$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} -4(\text{II})$

$\downarrow$ row-reduced echlon form

$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

solution to $Ax = 0$:

$\begin{cases} x_1 = -2x_3 + x_4 \\ x_2 = -x_4 \end{cases}$

let $x_3 = 0$, $x_4 = 1$

$x_1 = 1$, $x_2 = -1$ $\quad V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

let $x_3 = 1$, $x_4 = 0$

$\Rightarrow x_1 = -2$, $x_2 = 0$

$V_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

V_1, V_2 not multiple of each other

$\Downarrow$

lin. indep. & is basis to $N(A)$

Thus $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$W_1 \in W_1 \cup W_2$

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Problem 2. (5)

Let V be a vector space over a field F. Suppose that W_1 and W_2 are two subspaces, neither of them is contained in the other. Prove or disprove the following statements:

1 (a) $W_1 \cap W_2$ is a subspace;

2 (b) $W_1 \cup W_2$ is a subspace;

1 (c) $W_1 + W_2$ is a subspace, where $W_1 + W_2$ is defined to be the collection of elements of the form $z = x + y$ with $x \in W_1$ and $y \in W_2$.

1 (d) If $W_1 \cap W_2 = \{0\}$, any element z in $W_1 + W_2$ can be uniquely expressed as $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(a) ~~Yes~~ True

① let $\vec{v} \in W_1 \cap W_2$

consider $\lambda\vec{v}$, where $\lambda \in F$

Since $\vec{v} \in W_1$, W_1 is subspace

$\Rightarrow \lambda\vec{v} \in W_1$

Since $\vec{v} \in W_2$, W_2 is subspace

$\Rightarrow \lambda\vec{v} \in W_2$

$\Rightarrow \lambda\vec{v} \in W_1 \cap W_2$ ✓

② let $\vec{v}_1 \in W_1 \cap W_2$, $\vec{v}_2 \in W_1 \cap W_2$

Consider $\vec{v}_1 + \vec{v}_2$

Since $\vec{v}_1 \in W_1$, $\vec{v}_2 \in W_1$, W_1 is subspace

$\Rightarrow \vec{v}_1 + \vec{v}_2 \in W_1$

Since $\vec{v}_1 \in W_2$, $\vec{v}_2 \in W_2$, W_2 is subspace

$\Rightarrow k\vec{v}_1 + \vec{v}_2 \in W_2$

$\Rightarrow \vec{v}_1 + \vec{v}_2 \in W_1 \cap W_2$ ✓

③ Since ~~$0 \in W_1$~~

W_1, W_2 subspace

$\Rightarrow 0 \in W_1$, $0 \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2$ ✓

~~(b) True.~~

~~① let $\vec{v} \in W_1 \cup W_2$~~

~~consider $\lambda\vec{v}$~~

~~case a. $\vec{v} \in W_1$, W_1 subspace~~

~~$\Rightarrow \lambda\vec{v} \in W_1$~~

~~$\Rightarrow \lambda\vec{v} \in W_1 \cup W_2$ ✓~~

~~case b. $\vec{v} \in W_2$, W_2 subspace~~

(b) False

let $V = \mathbb{R}^2$,

$W_1 = \{(0, x) : x \in \mathbb{R}\}$

$W_2 = \{(x, 0) : x \in \mathbb{R}\}$

Problem 3. (5)

Let $V = P_4(\mathbb{R})$ and $W = P_3(\mathbb{R})$ be the vector spaces of real polynomials of degree less or equal to 4 and 3 respectively with the standard bases $\beta = \{1, x, x^2, x^3, x^4\}$ and $\gamma = \{1, x, x^2, x^3\}$ accordingly. Consider the linear transformation $T : V \to W$ and $U : W \to V$ given by $T(f) = f'$ and $U(g) = \int_0^x g$ respectively.

(i) Find the matrices $[T]_\beta^\gamma$ and $[U]_\gamma^\beta$.

(ii) Find the matrices $[U \circ T]_\beta$ and $[T \circ U]_\gamma$.

(iii) Find the matrix $[(U \circ T)^n]_\beta$. Here $(U \circ T)^n = (U \circ T) \circ (U \circ T) \circ \cdots \circ (U \circ T)$ is the n-fold composition of $U \circ T$.

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(i) $[T]_\beta^\gamma$

$T(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_\gamma$

$T(x) = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_\gamma$

$T(x^2) = 2x = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}_\gamma$

$T(x^3) = 3x^2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}_\gamma$

$T(x^4) = 4x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}_\gamma$

$[T]_\beta^\gamma = \begin{bmatrix} [T(1)]_\gamma & \cdots & [T(x^4)]_\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

$[U]_\gamma^\beta$

$U(1) = \int_0^x 1 = x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_\beta$

$U(x) = \int_0^x x = \frac{1}{2}x^2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}_\beta$

$U(x^2) = \int_0^x x^2 = \frac{1}{3}x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}_\beta$

$U(x^3) = \int_0^x x^3 = \frac{1}{4}x^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}_\beta$

$[U]_\beta^\gamma = \begin{bmatrix} [U(1)]_\beta & \cdots & [U(x^3)]_\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

Problem 4. (5)

Let $V = M_{22}(\mathbf{R})$ be the vector space of 2 by 2 matrices and $T : V \to V$ be a linear transformation defined by $T(X) = X + X^t$ for $X \in V$. Here X^t is the transpose of the matrix X.

(i) What are the null space $N(T)$ and range $R(T)$ of T?

(ii) Find a basis for $N(T)$ and $R(T)$ respectively. What are the nullity and rank of T?

(iii)What are the intersection $N(T) \cap R(T)$ and the sum $N(T) + R(T)$?

Here the sum W_1+W_2 of two subspaces W_1 and W_2 is defined to be the collection of elements of the form $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(i) null space:

let $X \in V$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbb{R}$

$T(x)$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$= \begin{bmatrix} 2a & c+b \\ c+b & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a = 0,\ d = 0,\ c = -b.$

Thus $N(T)$

$= \left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ ✓

(ii) basis for $N(T)$:

$\beta = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

① β is lin. indep since $|\beta| = 1$

② given $v \in N(T)$,

$v = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = a \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = a\beta,$

Thus β spans $N(T)$

$\Downarrow$

β is basis. ✓

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range:

let $x \in V$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$T(x)$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$= \begin{bmatrix} 2a & c+b \\ c+b & 2d \end{bmatrix} = \begin{bmatrix} e & g \\ g & f \end{bmatrix}$

$R(T)$

$= \left\{ \begin{bmatrix} a & c \\ c & b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

this show $R(T) \subseteq \{\}$

why "="?

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