MATH 115A Midterm I, Fall 2018

Name:

Justify All Your Answers

Problem 1. (5)

Let T be a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^3$ to itself defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$\begin{array}{cc} & \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ A = & \begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 12 & 6 & 9 \\ 4 & 14 & 8 & 10 \end{bmatrix} \end{array}.$$

and $\mathbf{x}$ is a column vector in $\mathbf{R}^4$.

(i) Find a basis of the null space of T.

(ii) Find a basis of the range of T.

(iii) What is nullity and rank of T?

2 pts $$A \longrightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 2 & -1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\Rightarrow$ (ii) $\left\{ v_1 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 4 \\ 12 \\ 14 \end{pmatrix} \right\}$ is a

1 pt basis for $\mathcal{R}(T)$.

(i) Basis for $N(T)$. $x_3 = 1, x_4 = 0 \Rightarrow x_1 = -2, x_2 = 0$

$\Rightarrow n_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad x_3 = 0, x_4 = 1 \Rightarrow x_1 = 1, x_2 = -1.$

2 pts

$\Rightarrow n_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \{n_1, n_2\}$ is a basis.

Problem 2. (5)

Let V be a vector space over a field F. Suppose that W_1 and W_2 are two subspaces, neither of them is contained in the other. Prove or disprove the following statements:

(a) $W_1 \cap W_2$ is a subspace;

(b) $W_1 \cup W_2$ is a subspace;

(c) $W_1 + W_2$ is a subspace, where $W_1 + W_2$ is defined to be the collection of elements of the form $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(d) If $W_1 \cap W_2 = \{0\}$, any element z in $W_1 + W_2$ can be uniquely expressed as $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(a) $W_1 \cap W_2$ is a subspace: $w_1, w_2 \in W_1 \cap W_2 \Leftrightarrow$

1 pt w_1 & $w_2 \in W_1$ & $W_2 \Rightarrow \forall c_1$ & $c_2 \in F$, $c_1 w_1 + c_2 w_2 \in W_1$ & W_2

$\therefore c_1 w_1 + c_2 w_2 \in W_1 \cap W_2$

(b) $W_1 \cup W_2$ is not a subspace in this case:

by the assumption $\exists w_1 \in W_1$ not in W_2 & $w_2 \in W_2$ not

2 pts in $W_1 \Rightarrow w_1 + w_2 \notin W_1$ otherwise $w_2 = (w_1 + w_2) - w_1 \in W_1$, a contradiction. Similarly $w_1 + w_2 \notin W_2 \Rightarrow w_1 + w_2 \notin W_1 \cup W_2$.

(c) $W_1 + W_2$ is a subsp: for $z_1 = x_1 + y_1$ & $z_2 = x_2 + y_2 \in W_1 + W_2$

& $\forall c_1, c_2 \in F$, $c_1 z_1 + c_2 z_2 = \underbrace{(c_1 x_1 + c_2 x_2)}_{\in W_1} + \underbrace{(c_1 y_1 + c_2 y_2)}_{\in W_2}$

1 pt $\in W_1 + W_2$.

(d) If $z = x_1 + y_1 = x_2 + y_2$ w/ $x_i \in W_1$, $y_i \in W_2$, $i = 1, 2$

$\Rightarrow x_1 - x_2 = y_2 - y_1 \in W_1 \cap W_2 = \{0\}$

1 pt $\Rightarrow x_1 = x_2$ & $y_1 = y_2 \Rightarrow$ (d) is true.

1 pt (iii) $(U\circ T)^2(\beta) = ((U\circ T)^2(1), (U\circ T)^2(x), (U\circ T)^2(x^2), (U\circ T)^2(x^3), (U\circ T)^2(x^4))$

$= (U\circ T(0), U\circ T(x), U\circ T(x^2), U\circ T(x^3), U\circ T(x^4)) = (0, x, x^2, x^3, x^4)$

$= (U\circ T)(\beta) \Rightarrow (U\circ T)^2 = U\circ T$. Inductively $(U\circ T)^n = U\circ T$

$\therefore [(U\circ T)^n]_\beta = [U\circ T]_\beta$

Problem 3. (5)

Let $V = P_4(\mathbb{R})$ and $W = P_3(\mathbb{R})$ be the vector spaces of real polynomials of degree less or equal to 4 and 3 respectively with the standard bases $\beta = \{1, x, x^2, x^3, x^4\}$ and $\gamma = \{1, x, x^2, x^3\}$ accordingly. Consider the linear transformation $T : V \to W$ and $U : W \to V$ given by $T(f) = f'$ and $U(g) = \int_0^x g$ respectively.

(i) Find the matrices $[T]_\beta^\gamma$ and $[U]_\gamma^\beta$.

(ii) Find the matrices $[U \circ T]_\beta$ and $[T \circ U]_\gamma$.

(iii) Find the matrix $[(U\circ T)^n]_\beta$. Here $(U\circ T)^n = (U\circ T)\circ(U\circ T)\circ\cdots\circ(U\circ T)$ is the n-fold composition of $U \circ T$.

(i) $T(\beta) = (T(1), T(x), T(x^2), T(x^3), T(x^4))$

1 pt $= (0, 1, 2x, 3x^2, 4x^3) = (\underbrace{1, x, x^2, x^3}_{\gamma}) \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}}_{[T]_\beta^\gamma}$

$U(\gamma) = (U(1), U(x), U(x^2), U(x^3))$

1 pt $= (x, \frac{x^2}{2}, \frac{x^3}{3}, \frac{x^4}{4}) = (1, x, x^2, x^3, x^4) \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}}_{[U]_\gamma^\beta}$

(ii) $(U\circ T)(\beta) = (U\circ T(1), U\circ T(x), U\circ T(x^2), U\circ T(x^3), U\circ T(x^4))$

1 pt $= (0, x, x^2, x^3, x^4) = (\underbrace{1, x, x^2, x^3, x^4}_{\beta}) \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[U\circ T]_\beta}$

$(T\circ U)(\gamma) = (T\circ U(1), T\circ U(x), (T\circ U)(x^2), (T\circ U)(x^3))$

1 pt $= (1, x, x^2, x^3) = (\underbrace{1, x, x^2, x^3}_{\gamma}) \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{[T\circ U]_\gamma}$

(iii) $N(T) \cap R(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

1.5 pts

$N(T) + R(T) = V$ pf: $\forall X \in V$.

$$X = \underbrace{\frac{X + X^t}{2}}_{① \; N(T)} + \underbrace{\frac{X - X^t}{2}}_{① \; R(T)}$$

Problem 4. (5)

Let $V = M_{22}(\mathbb{R})$ be the vector space of 2 by 2 matrices and $T : V \to V$ be a linear transformation defined by $T(X) = X - X^t$ for $X \in V$. Here X^t is the transpose of the matrix X.

(i) What are the null space $N(T)$ and range $R(T)$ of T?

(ii) Find a basis for $N(T)$ and $R(T)$ respectively. What are the nullity and rank of T?

(iii)What are the interesction $N(T) \cap R(T)$ and the sum $N(T) + R(T)$?

Here the sum $W_1 + W_2$ of two subspaces W_1 and W_2 is defined to be the collection of elements of the form $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(i) $N(T) = \{ X \mid X^t = X \}$

$=$ the subsp of all 2×2 symm. matrices.

$R(T) =$ the subsp of all 2×2 skew symm matrices

pf: If A is skew symm $\Rightarrow A^t = -A$

2 pts

$\therefore A = T(A/2) \; \left(= \frac{A - A^t}{2} = A \right)$

i.e. $A \in R(T)$. conversely for

any $y \in R(T)$, $y = T(x)$ for some $x \in V$.

Then $y^t = (T(x))^t = (x - x^t)^t = x^t - x = -y$

so that y is skew symm. □

(ii) Basis for $N(T)$: $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

1.5 pts

$\therefore$ Nullity $= 3$.

Basis for $R(T)$: $\alpha = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. $Rk(T) = 1$.