ANSWER SHEET

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Problem 1 (25 points) Give specific examples of each of the following four items (You do not need to justify):

a. Four 6-dimensional vector spaces V_1, V_2, V_3, V_4 over a field F containing no subspace of any of the others and a 3-dimensional subspace W_i of V_i for $i = 1, 2, 3, 4$.

$V_1 =$ $P_5[t]$

all polynomials of degree 5

$W_1 =$ $P_2[t]$

all polynomials of degree 2

$V_2 =$ M_{23}

$W_2 =$ subspace spanned by

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$V_3 =$ $\mathbb{R}^6$

$W_3 =$ subspace spanned by

$(0,0,0,0,0,1)$

$(0,0,0,0,1,0)$

$(0,0,0,1,0,0)$

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$V_4 =$ $P_5(t)$

all polynomial funcs of degree 5 / $\mathbb{R}$

$W_4 =$ $P_2(t)$

all polynomial funcs of degree 2 / $\mathbb{R}$

b. An infinite dimension vector space V over a field F, an infinite dimensional subspace X of V but not V and three 3-dimensional subspaces W_1, W_2, and W_3 of V such that $W_1 + W_2 + W_3$ is 9-dimensional.

$V =$ $\text{Fcn}(0,1)$

$X =$ $C(0,1)$
↑
continuous

$W_1 =$ subspace spanned by $\{e^x, e^{2x}, e^{3x}\}$

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$W_2 =$ . . . . $\{\sin x, \sin 2x, \sin 3x\}$

$W_3 =$ . . . . $\{x, x^2, x^3\}$

c. Two continuous real-valued functions on the unit interval $[0, 1]$ that are not polynomial functions but are linearly independent.

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$f(x) = \sin x$, $g(x) = e^x$.

d. A linear transformation $T : C^1(0, 1) \to C^3(0, 1)$ where $C^n(0, 1)$ is the vector space of real valued functions $f : [0, 1] \to \mathbf{R}$ that have continuous derivatives of order n on the closed interval $[0, 1]$.

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$T : C^1(0,1) \to C^3(0,1)$;

$f(x) \longmapsto \int_0^x \int_0^x f(t)\, dt\, ds$

Problem 2 (25 points) Do all of the following (there are two parts):

a. Accurately state the full content of the following (named) theorems that we have proven in class.

Toss In Theorem

V a vs $/F$, $\emptyset \neq S \subset V$ lin. indep.

If $\exists\, v \in V \ni v \notin \text{span}(S)$,

then $S \cup \{v\}$ is lin. indep.

Toss Out Theorem

V a vs$/F$, $\emptyset \neq S \subset V$, $V = \text{Span}(S)$

then $\exists\, T \subset S$, ~~[illegible]~~.

T a basis of V.

Replacement Theorem

~~[illegible]~~

V a fdvs$/F$, $\mathcal{B} = \{v_1, \dots, v_n\}$ a basis.

if $\exists\, v \in V$, $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$, ~~[illegible]~~ where $a_i \neq 0$,

then $\mathcal{B}' = \{v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n\}$ is a basis.

Dimension Theorem

$T : V \to W$ linear, V ~~fdv~~ fdvs$/F$ W vs$/F$

$\ker T$, $\text{im}\, T$ fd

$\dim V = \dim \ker T + \dim \text{Im}\, T$.

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b. Give a consequence (e.g., corollary or example or application) of three of the theorems stated in (a). (You do not need to justify.)

First Consequence and of which theorem

~~Dimension thm: if $\ker T = 0$, then $\dim V = \dim W$,~~

~~and T must be one-to-one.~~

Dimension: $T: \mathbb{R}^2 \to \mathbb{R}^2$ via $(a,b) \mapsto (a+b, 2a+2b)$

has ~~image~~ image of dimension 1 because

$(a+b, 2a+2b) = 0 \Rightarrow a+b=0 \Rightarrow \ker(T) = \text{span}\{(-1,1)\}$

$\Rightarrow \dim \text{Im}(T) = \dim(V) - \dim\ker(T) = 2-1 = 1$.

Second Consequence and of which theorem

Replacement Thm: $V = \mathbb{R}^3$. $\mathcal{B} = \{e_1, e_2, e_3\}$,

then $\mathcal{C} = \{(1,1,0), (0,1,0), (0,0,2)\}$

is a basis.

Third Consequence and of which theorem

Tossin:

$V = \mathbb{R}^3$ $S = \{e_1, e_2\}$, $v = (0,0,1) \notin \text{span}(S)$

$\therefore$ $S \cup \{v\} = \{e_1, e_2, e_3\}$ is lin. indep.

Problem 3 (25 points) Define two linear transformations as follows:

$$T : \mathbf{R}^3 \to \mathbf{R}^4 \quad \text{is given by} \quad T(a,b,c) = (a+b,\ a-b, a, c)$$

and

$$S : \mathbf{R}^4 \to \mathbf{R}[t] \quad \text{is given by} \quad S(a,b,c,d) = (a+b) + ct + dt^3.$$

Find the dimensions of the kernel (null space) and the image (range) of all of the following: S, T, $S \circ T$. Label them carefully and give brief justifications (if you need more room continue on the back or another sheet).

The nullity $\dim(\ker(T)) =$ 0 .

Reason solve for $\begin{cases} a+b=0 \\ a-b=0 \\ a=0 \\ c=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases} \Rightarrow \ker(T) = 0.$

The rank $\dim(\text{im}(T)) =$ 3 .

Reason $\dim(V) = 3 = \dim im(T) + \underbrace{\dim\ker(T)}_{0}.$

$\therefore \dim im(T) = 3.$

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The nullity $\dim(\ker(S)) =$ 1

Reason $\begin{cases} a+b=0 \\ c=0 \\ d=0 \end{cases} \Rightarrow \ker(T) = \text{Span}(\{(-1,1,0,0)\})$

The rank $\dim(\text{im}(S)) =$ 3.

Reason $\dim(R^4) = 4 = \underbrace{\dim im(S)}_{1} + \dim\ker(S)$

$\therefore$ ~~dim(ker(S))~~ $= 3$.
dim(im(S))

The nullity $\dim\left(\ker(S \circ T)\right) =$ 1

Reason

$\begin{cases} a_s + b_s = 0 \\ c_s = 0 \\ d_s = 0 \end{cases} \Rightarrow \begin{cases} 2a_T = 0 \\ a_T = 0 \\ c_T = 0 \end{cases} \Rightarrow \ker(S \circ T) = \operatorname{span}(\{(0,1,0)\})$

The rank $\dim\left(\operatorname{im}(S \circ T)\right) =$ 2

Reason

$S \circ T: \mathbb{R}^3 \to \mathbb{R}(t)$

$\dim(\operatorname{im}(S \circ T)) = \dim(\mathbb{R}^3) - \dim\ker(S \circ T)$

$= 3 - 1 = 2.$

Problem 4 (25 points) Prove the Toss In Theorem.

[Note there is a problem five which is to redo the test at home and turn it in next Monday.]

Given V as VS/F, $0 \notin S \subset V$ lin independent,

NTS: $S \cup \{v\}$ is lin. indep.

$v \in V$, $v \notin \operatorname{span}(S)$. $S = \{v_1, \dots, v_n\}$ not finite

Now, suppose $T = S \cup \{v\}$ is lin. dep.

Then $\exists\ \alpha, \alpha_1, \dots, \alpha_n$ not all zero $\ni$

$\alpha v + \alpha_1 v_1 + \dots + \alpha_n v_n = 0$

$\Rightarrow$ ① $\alpha = 0$ then $\alpha_1, \dots, \alpha_n$ not all zero

$\Rightarrow S = \{v_1, \dots, v_n\}$ is lin dep.

$\Rightarrow$ contradiction.

② $\alpha \neq 0$. then $\exists\ \alpha^{-1} \in F$, so

$v = -\alpha^{-1}\alpha_1 v_1 - \alpha^{-1}\alpha_2 v_2 - \dots - \alpha^{-1}\alpha_n v_n$.

$\Rightarrow v \in \operatorname{span}(S) \Rightarrow$ contradiction.

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$\therefore$ Proof by contradiction. $\square$.