MATH 115A/6. EXAM I. 26 OCTOBER 2015. R. De Sapio

NAME [redacted]
Last (Family), First (Given)

STUDENT I.D. Nº [redacted]

SIGNATURE [redacted]

Question (Value)	Score
1 (	25
2 (	23
3 (	25
4 (	20
Total (100)	98

NOTE. The field F is either $\mathbb{R}$ or $\mathbb{C}$.

1. By a direct computation using the definition of span show that the vectors (2,1) and (1,1) span $\mathbb{R}^2$.

$S = \{S_1, S_2, \ldots S_n\}$

Def: A set of vectors S spans a v.s. V iff for all $v \in V$, there exists $a_1 \ldots a_n \in \mathbb{F}$ s.t.

$a_1 S_1 + a_2 S_2 + \ldots + a_n S_n = v$. In other words every $v \in V$ is equal to a linear combination of vectors in S.

We take an arbitrary vector $x \in V$, where $x = (x_1, x_2)$

We show that the above definition holds for $S = \{(2,1), (1,1)\}$.

$a_1(2,1) + a_2(1,1) = (x_1, x_2)$

$2a_1 + a_2 = x_1$

$a_1 + a_2 = x_2$

$a_2 = x_1 - 2a_1$

$a_1 + x_1 - 2a_1 = x_2$

$-a_1 = x_2 - x_1$

$\boxed{a_1 = x_1 - x_2}$

$x_1 - x_2 + a_2 = x_2$

$\boxed{a_2 = -x_1 + 2x_2}$

25 ✓

For all $x \in \mathbb{R}^2$, we let $a_1 = x_1 - x_2$ and $a_2 = -x_1 + 2x_2$ in order to satisfy $a_1(2,1) + a_2(1,1) = (x_1, x_2)$. Therefore (2,1) and (1,1) span $\mathbb{R}^2$, since every $x \in \mathbb{R}^2$ is equal to a linear combination of the vectors (2,1), (1,1)

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2. Let u, v, w be linearly independent vectors in a vector space V over a field F. Prove that the three vectors $u+v$, $u+w$, $v+w$ are also linearly independent by using the definition of linear independence.

Def: Linear Independence: Any set of vectors which is NOT linearly dependent

Def: Linear dependence: Any nonempty set of vectors $\{v_1 \dots v_n\}$ such that there exist scalars $a_1 \dots a_n \in F$ s.t.

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

and there exists at least one nonzero scalar a_i.

① $a_1 u + a_2 v + a_3 w = 0$ is only true when $a_1, a_2, a_3 = 0$ by definition of linear independence.

Now we want to prove $u+v$, $u+w$, $v+w$ is linearly independent

Proof by Contradiction:

Suppose $u+v$, $u+w$, $v+w$ is linearly dependent. This means

$$a_1(u+v) + a_2(u+w) + a_3(v+w) = 0$$

holds for scalars (not all zero) a_1, a_2, a_3. However, we rearrange the equation to see:

3

② $(a_1+a_2)u + (a_1+a_3)v + (a_2+a_3)u = 0$

so $a_1+a_2 = a_1+a_3 = a_1+a_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0$

This means that ~~there~~ must exist some $a_i + a_j$ term which is nonzero, which is a contradiction to statement ①, which says the equation only holds when all coefficients are zero.

Therefore, $u+v$, $u+w$, and $v+w$ must be linearly independent

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3. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(a_1, a_2, a_3) = (a_1 - a_2,\ 2a_3,\ a_1 - a_2)$. Find a basis for the null space $N(T)$. What is the dimension of $N(T)$? What is the rank of T?

$a_1 - a_2 = 0$
$2a_3 = 0$
$a_1 - a_2 = 0$

$a_3 = 0$
$a_1 = a_2$

$(a_1, 0, a_1)$

−5

$\{(1,1,0)\}$

$a_3 = 0$?

A) Basis for $N(T)$. $\{(1, 0, 1)\}$

$\{(1, 0, 1)\}$ spans all vectors of the form $(a_1, 0, a_1)$

B) $\text{Dim}(N(T)) = 1$ ✓

The dimension is defined as the number of vectors in the basis of a space

C) $\text{Rank}(T) = 2$ ✓

By Rank Nullity theorem, we know $\text{Rank}(T) + \text{Nullity}(T) = \text{Dim}(V)$. $\text{Dim}(V) = 3$, since the dimension of $\mathbb{R}^3$ is 3, and $\text{Nullity}(T)$ is 1 as proved in B). Then, we solve for $\text{Rank}(T)$

$\text{Rank}(T) + 1 = 3$

$\text{Rank}(T) = 2$

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...e 7 questions on the test, each on a s... ...oose any subset to answer. If y... ...n that page. Each probl...

4. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by $T(1,1) = (2,2)$ and $T(-1,1) = (-3,3)$ on the basis $B' \{(1,1), (-1,1)\}$ for $\mathbb{R}^2$. Find the matrix of T relative to the standard basis $\{(1,0),(0,1)\}$. B

$B_1 = \frac{1}{2}B_1' - \frac{1}{2}B_2'$

$B_2 = \frac{1}{2}B_1' + \frac{1}{2}B_2'$

$B_1' = B_1 + B_2$

$B_2' = -B_1 + B_2$

Q = change of coordinate Matrix from B to B' $= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Q^{-1} change of coordinate Matrix from B' to B $= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$[T]_B = [Q^{-1}]_{B'}^{B} [T]_{B'} [Q]_B^{B'}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5/2 & -\frac{1}{2} \\ -\frac{1}{2} & 5/2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6/4 & 4/4 \\ -6/4 & 4/4 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 1 \\ -3/2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$[T]_B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$Ax + By = (2,2)$

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$A + B = 2$

$C + D = 2$

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

$-A + B = -3$

$-C + D = 3$

$A = \frac{5}{2}$ $B = -\frac{1}{2}$

$C = -\frac{1}{2}$ $D = \frac{5}{2}$

$[T]_{B'} = \begin{bmatrix} 5/2 & -\frac{1}{2} \\ -\frac{1}{2} & 5/2 \end{bmatrix}$ ✓