Math 115A - Winter 2019
Exam 2 - Solutions

Full Name: ______________________________

UID: ______________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

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1. (10 points) True or False: Prove or disprove the following statements.

   (a) Let $S, T : V \to V$ be linear operators on a finite-dimensional vector space. Assume that S and T commute, i.e. that $ST = TS$. If v is an eigenvector of T such that $S(v) \neq 0$, then $S(v)$ is also an eigenvector of T.

   (b) Let A be an $n \times n$ matrix with v and w two eigenvectors of A. Then $v + w$ is an eigenvector of A.

Solution:

(a) **True.**

Proof. Let $v \in V$ be an eigenvector of T with eigenvalue λ and assume that $S(v) \neq 0$. Then by commutativity

$$T(S(v)) = TS(v) = ST(v) = S(\lambda v).$$

But S is linear, so $S(\lambda v) = \lambda S(v)$. Thus $S(v)$ is an eigenvector for T with eigenvalue λ. □

(b) **False.** In general, v and w may be two eigenvectors with distinct eigenvalues. In particular, suppose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

so that $v = e_1 = (1, 0)$ and $w = e_2 = (0, 1)$ are eigenvectors with eigenvalues $\lambda = 1$ and $\mu = 2$, respectively. Then $v + w = (1, 1)$ but

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and $(1, 2)$ is not a scalar multiple of $v + w = (1, 1)$.

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2. (10 points) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V.

(a) Suppose $T^n = 0$ for some n. Prove that the only eigenvalue of T is zero.

(b) Show that $T^2 = 0$ if and only if $\operatorname{im} T \subseteq \ker T$.

Solution:

(a) *Proof.* Suppose that λ is an eigenvalue of T. Then there exists a nonzero vector $v \in V$ such that $T(v) = \lambda v$. Now applying T^n to v we have

$$0 = T^n(v) = T(T(\cdots T(v))) = \lambda^n v.$$

Since $v \neq 0$, it must be that $\lambda^n = 0$. But then $\lambda = 0$. □

(b) *Proof.* ($\Longrightarrow$) Suppose that $T^2 = 0$. Let $w \in \operatorname{im} T$. Then there exists some $v \in V$ such that $T(v) = w$. Now since $T^2 = 0$ we have

$$0 = T^2(v) = T(T(v)) = T(w),$$

so $w \in \ker T$. Thus $\operatorname{im} T \subseteq \ker T$.

($\Longleftarrow$) Now suppose $\operatorname{im} T \subseteq \ker T$. Let $v \in V$ be arbitrary. If we call $w = T(v)$, since $\operatorname{im} T \subseteq \ker T$ we know $T(w) = 0$. Now apply T^2 to v to get

$$T^2(v) = T(T(v)) = T(w) = 0.$$

Since $v \in V$ was arbitrary, $T^2(v) = 0$ for all v. Hence $T^2 = 0$. □

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3. (10 points) True or False: Prove or disprove the following statements.

Consider the linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $T(f(x)) = f'(x) + f(0)$.

(a) The linear operator T is one-to-one.

(b) The linear operator T is onto.

Solution:

(a) **False.** We will show T is not one-to-one by showing that $\ker T \neq \{0\}$. Consider the polynomial $f(x) = x - 1 \in \mathbb{R}[x]$. Now compute

$$T(f(x)) = f'(x) + f(0) = 1 + (-1) = 0$$

so $f(x) = x - 1 \in \ker T$ but $f(x) \neq 0$. Hence T is not one-to-one.

(b) **True.**

Proof. It suffices to show that every element in the standard basis for $\mathbb{R}[x]$ given by $\beta = \{1, x, x^2, \dots\}$ is contained $\operatorname{im} T$. Notice that

$$x^n = T\left(\frac{x^{n+1}}{n+1}\right)$$

for all $n \geq 0$. So every basis element of β is in $\operatorname{im} T$ and T is onto. □

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4. (10 points) Let $V = M_{2\times 2}(\mathbb{R})$ and $W = P_3(\mathbb{R})$. Let

$$\beta = \left\{ w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, w_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and}$$
$$\gamma = \{1, x, x^2, x^3\}$$

be the standard bases. Consider the linear map $T : V \to W$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ax^3 + (c - b + 2d)x^2 + 2(c+d)x + (c+d).$$

(a) Determine $[T]_\beta^\gamma$.

(b) Prove that although $V \cong W$, the map T is not an isomorphism. (*Hint:* The proof that $V \cong W$ should be one line.)

Solution:

(a) We need to express $T(w_1), T(w_2), T(w_3), T(w_4)$ in the γ basis. So we compute

$$T(w_1) = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = x^3$$
$$T(w_2) = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -x^2$$
$$T(w_3) = T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = x^2 + 2x + 1$$
$$T(w_4) = T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2x^2 + 2x + 1$$

Collecting up the coefficients we have

$$\boxed{[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -1 & 1 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}}.$$

(b) *Proof.* Observe that $V \cong W$ by the dimension theorem, since both V and W have dimension 4. So there exists an isomorphism $V \to W$, however T is not an isomorphism. We can see clearly by explicit computation of the determinant, or because the first and second rows of the matrix above are scalar multiples of each other, that $\det[T]_\beta^\gamma = 0$. Thus the matrix $[T]_\beta^\gamma$ is not invertible. Hence the linear operator T is not invertible and so not an isomorphism. $\square$

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