Math 115A - Winter 2019
Exam 1 - Solutions

Full Name: ____________________________________

UID: ____________________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	10	
2	15	
3	5	
4	10	
Total:	40	

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Recall for a matrix $A \in M_{2\times 2}(\mathbb{F})$ the *trace* of A and the *determinant* of A are given by

$$\operatorname{tr}(A) = \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d, \qquad \det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

1. (10 points) True or False: Prove or disprove the following statements.

   (a) The set $W = \{A \in M_{2\times 2}(\mathbb{F}) \mid \operatorname{tr}(A) = 0\}$ is a subspace of $M_{2\times 2}(\mathbb{F})$.

   (b) The set $W = \{A \in M_{2\times 2}(\mathbb{F}) \mid \det(A) = 0\}$ is a subspace of $M_{2\times 2}(\mathbb{F})$.

Solution:

(a) **True.**

Proof. In order to show W is a subspace we need to show it is closed under addition and scalar multiplication, and that W contains zero. Suppose we have two arbitrary elements $A, B \in W$ so

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

with $a + d = 0$ and $w + z = 0$, and $\lambda \in \mathbb{F}$. Then

$$A + B = \begin{pmatrix} a + w & b + x \\ c + y & d + z \end{pmatrix}$$

has trace $(a + w) + (d + z) = (a + d) + (w + z) = 0 + 0 = 0$. So W is closed under addition. Now for scalar multiplication consider

$$\lambda A = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix},$$

which has trace $\lambda a + \lambda d = \lambda(a + d) = \lambda 0 = 0$. So W is closed under scalar multiplication. The zero matrix has all zero entries so $\operatorname{tr}(0) = 0 + 0 = 0$, and so $0 \in W$. Thus W is a subspace of $M_{2\times 2}(\mathbb{F})$. □

An alternative solution: show that $tr : M_{2\times 2}(\mathbb{F}) \to \mathbb{F}$ is a linear operator and that $W = \ker T$.

(b) **False.**

Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in W.$$

Then

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has determinant 1, so $A + B$ is not in W.

2. (15 points) Let S_1 and S_2 be lists of vectors in a vector space V. Let $W_1 = \operatorname{span}(S_1 \cap S_2)$ and $W_2 = \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$

(a) Prove that W_1 is a subset of W_2.

(b) Given an example of S_1 and S_2 for which W_1 and W_2 are equal.

(c) Given an example of S_1 and S_2 for which W_1 and W_2 are not equal.

Solution:

(a) *Proof.* Suppose $v \in W_1 = \operatorname{span}(S_1 \cap S_2)$ is an arbitrary element. Then v is a linear combination of some vectors in S_1 and S_2, with $v = \lambda_1 v_1 + \cdots \lambda_n v_n$ for some $\lambda_i \in \mathbb{F}$ and $v_i \in S_1 \cap S_2$. Since each $v_i \in S_1$, the vector v is a linear combination of elements from S_1 and so $v \in \operatorname{span}(S_1)$. Similarly, since each $v_i \in S_2$, the vector v is a linear combination of elements from S_2, which implies $v \in \operatorname{span}(S_2)$. So $v \in \operatorname{span}(S_1) \cap \operatorname{span}(S_2) = W_2$. Thus $W_1 \subseteq W_2$. □

(b) There are many possible answers. Let $V = \mathbb{R}^2$ and take $S_1 = \{(1,0)\}$ and $S_2 = \{(0,1)\}$. Then $S_1 \cap S_2$ is empty and $W_1 = \operatorname{span}(S_1 \cap S_2) = \{0\}$. On the other hand, $\operatorname{span}(S_1)$ is the x-axis and $\operatorname{span}(S_2)$ is the y-axis. The only point on both axes is the origin so also $W_2 = \operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \{0\}$.

(c) There are again many possible answers. Let $V = \mathbb{R}^2$ and take $S_1 = \{(1,0),(0,1)\}$ and $S_2 = \{(1,1)\}$. Again $S_1 \cap S_2$ is empty so $W_1 = \operatorname{span}(S_1 \cap S_2) = \{0\}$. But now $\operatorname{span}(S_1) = \mathbb{R}^2$ and $\operatorname{span}(S_2)$ is the diagonal in $\mathbb{R}^2$, i.e. $\{(x,x) \mid x \in \mathbb{R}\}$. So $W_2 = \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$ is also the diagonal and $W_1 \neq W_2$.

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3. (5 points) True or False: Prove or disprove the following statement.

Let $V = \mathbb{R}^2$ and let $W_1 = \text{span}\{(2,1)\}$ and $W_2 = \text{span}\{(0,1)\}$. Then $V = W_1 \oplus W_2$.

Solution: True.

Proof. Since W_1 and W_2 are subspaces and $\mathbb{R}^2$ is closed under addition, it follows that $W_1 + W_2 \subseteq \mathbb{R}^2$. Let $(x,y) \in \mathbb{R}^2$ be arbitrary. Then we can write (x,y) as a linear combination of $(2,1)$ and $(0,1)$ since

$$a(2,1) + b(0,1) = (x,y)$$

if we let $a = \frac{x}{2}$ and $b = y - \frac{x}{2}$. So $(x,y) \in W_1 + W_2$ and $V \subseteq W_1 + W_2$. Now suppose we have $(x,y) \in W_1 \cap W_2$. Then $(x,y) \in W_1$ and $(x,y) \in W_2$. Since $W_1 = \text{span}\{(2,1)\}$ we can write $(x,y) = a(2,1) = (2a,a)$. But then $W_2 = \text{span}\{(0,1)\}$ so we can also write $(x,y) = b(0,1) = (0,b)$. The second equation gives $x = 0$ and then the first equation gives $2a = 0$ so $a = 0$ and also $y = 0$. Hence $(x,y) = (0,0)$ and so $W_1 \cap W_2 = \{0\}$. Thus $\mathbb{R}^2 = W_1 \oplus W_2$. □

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4. (10 points) Consider the function $T : M_{2\times 2}(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ defined by $T(M) = EM - ME$ where E is the matrix $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(a) Prove that T is a linear transformation.

(b) Find the rank and nullity of T.

Solution:

(a) *Proof.* Let $M, N \in M_{2\times 2}(\mathbb{F})$. Then T distributes over sums because

$$\begin{aligned} T(M+N) &= E(M+N) - (M+N)E \\ &= EM + EN - ME - NE \\ &= (EM - ME) + (EN - NE) \\ &= T(M) + T(N). \end{aligned}$$

For scalar multiplication, we check

$$T(\lambda M) = E(\lambda M) - (\lambda M)E = \lambda EM - \lambda ME = \lambda(EM - ME) = \lambda T(M).$$

So T is a linear transformation. □

(b) First we compute $T(M)$ explicitly for a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{aligned} T(M) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \\ &= \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}. \end{aligned}$$

If $M \in \ker T$ then $T(M) = 0$, so we must have $a = d$ and $c = 0$. So

$$\ker T = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{F} \right\},$$

which has a basis given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

So $\dim(\ker T) = 2$ and $\dim M_{2\times 2}(\mathbb{F}) = 4$, by the Rank-Nullity Theorem we have $\dim(\operatorname{im} T) = \dim M_{2\times 2}(\mathbb{F}) - \dim(\ker T) = 2$.

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