Math 115A Quiz 3

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Problem 1. *Honor Statement.*

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Problem 2.

(a) *Does the set* $\{(1,1,4,5),(8,2,0,0),(-1,6,2,3)\}$ *generate* $\mathbb{R}^4$*? Justify your answer.*

Solution. No, because the set only has 3 vectors and $\mathbb{R}^4$ has dimension 4. Any generating set must contain at least 4 vectors. □

(b) *Is the set* $\{1+x^2, 1+x, x-x^2, 1\}$ *linearly independent in* $P_2(\mathbb{R})$*? Justify your answer.*

Solution. No, because the set has 4 vectors and $P_2(\mathbb{R})$ only has dimension 3. Any linearly independent set must contain at most 3 vectors. □

(c) *Does the set* $\left\{\begin{pmatrix}1 & 5\\ -2 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}, \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}\right\}$ *span the complex vector space* $M_{2\times 2}(\mathbb{C})$*? Justify your answer.*

Solution. No, because the set only has 3 vectors and $M_{2\times 2}(\mathbb{C})$ has dimension 4. Any spanning set must contain at least 4 vectors. □

Problem 3. *Let* V *be a vector space and let* $v_1, \ldots, v_k, v \in V$*. Define* $W_1 = \operatorname{span}\{v_1, \ldots, v_k\}$ *and* $W_2 = \operatorname{span}\{v_1, \ldots, v_k, v\}$*. Prove that* v *can be written as a linear combination of the vectors* $v_1, \ldots, v_k$ *if and only if* $\dim(W_1) = \dim(W_2)$*.*

Proof. Notice that W_1 is a subset of W_2. In particular, let $u = a_1v_1 + \cdots + a_kv_k$ be a vector in W_1 for $a_1, \ldots, a_k \in \mathbb{F}$. Then u is also in W_2 as the linear combination

$$a_1v_1 + \cdots + a_kv_k + 0v$$

where $0 \in \mathbb{F}$ is the additive scalar identity.

We first prove in the forward direction. Suppose v can be written as a linear combination of the vectors $v_1, \ldots, v_k$, specifically $b_1v_1 + \cdots + b_kv_k$ for $b_1, \ldots, b_k \in \mathbb{F}$. We want to prove that W_2 is a subset of W_1. Any vector $w \in W_2$ can be written as a linear combination

$$c_1v_1 + \cdots + c_kv_k + cv$$

for $c_1, \ldots, c_k \in \mathbb{F}$.

Plugging in our expression for v:

$$c_1v_1 + \cdots + c_kv_k + c(b_1v_1 + \cdots + b_kv_k) = (c_1 + cb_1)v_1 + \cdots + (c_k + cb_k)v_k$$

Hence w is a linear combination of $v_1, \dots, v_k$ and thus in W_1. Since both subspaces are subsets of each other, they are equal. It follows that $\dim(W_1) = \dim(W_2)$.

Next, we prove in the backward direction. Suppose $\dim(W_1) = \dim(W_2)$. Since W_1 is a subset of W_2, which is itself a vector space, we can say that W_1 is a subspace of W_2. From Theorem 1.11, we have that $W_1 = W_2$; in particular, $v \in W_2$ must also be in W_1. By definition of span, v is a linear combination of $v_1, \dots, v_k$. □

Problem 4. *Let V be a vector space with finite dimensional subspaces W_1 and W_2. Prove that* $\dim(W_1 \cap W_2) \leq \dim(W_1)$.

Proof. From Theorem 1.4, we have that $W_1 \cap W_2$ is a subspace of V, so $\dim(W_1 \cap W_2)$ is well-defined. $W_1 \cap W_2$ is furthermore a subset of W_1, which is itself a vector space, so $W_1 \cap W_2$ is a subspace of W_1. From Theorem 1.11, we have that $\dim(W_1 \cap W_2) \leq \dim(W_1)$. □

Problem 5. *Let U denote the subspace of $M_{3\times3}(\mathbb{F})$ consisting of upper triangular 3×3 matrices. Find a basis for U (make sure to prove your choice is actually a basis). What is* $\dim(U)$*?*

Solution. Let $0, 1$ be the additive and multiplicative identities of $\mathbb{F}$ respectively. A basis for U is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

To see why, let $v_1, \dots, v_6$ be the individual matrices as ordered above. Notice that the upper triangular matrices of $M_{3\times3}(\mathbb{F})$ are defined as

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix}$$

for arbitrary field elements $a_1, \dots, a_6 \in \mathbb{F}$. This is also the linear combination $a_1v_1 + \cdots + a_6v_6$. Furthermore, this is unique since each scalar influences a separate element of the matrix. Hence $\{v_1, \dots, v_6\}$ is a basis of U. The size of a vector space's basis is its dimension, so $\dim(U) = 6$. □

Problem 6. *Suppose T is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T((2, 6)) = (2, -2)$ and $T((1, 3)) = (1, -5)$. Is it possible that T is a linear map? Why or why not?*

Proof. It is impossible for T to be a linear map. As a linear transformation, $T(cx) = cT(x)$ must hold for all $x \in V, c \in \mathbb{F}$. Notice that $(2, 6) = 2 \cdot (1, 3)$, yet $(2, -2) \neq 2 \cdot (1, -5)$. Hence T does not satisfy the requirement. □