Math 115A Quiz 2

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Problem 1. *Honor Statement.*

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Problem 2. *Let V be a vector space over a field $\mathbb{F}$. Prove that a subset W is a subspace if and only if $W \neq \emptyset$ and for all $a, b \in \mathbb{F}$ and $x, y \in W$, the linear combination $ax + by$ is also in W.*

Proof. Let W be a subset of V satisfying the properties above. Since W is non-empty, we can take vectors $x, y \in W$ (not necessarily distinct). Notice that the zero vector $\mathbf{0} \in V$ is a linear combination of x and y, specifically $0x + 0y = \mathbf{0} + \mathbf{0} = \mathbf{0}$ where $0 \in \mathbb{F}$ is the field's additive identity. Hence W contains the zero vector.

Next, for any scalar $a \in \mathbb{F}$, ax is a linear combination of x and y, specifically $ax + 0y = ax + \mathbf{0} = ax$. Hence W is closed under scalar multiplication.

Finally, $x + y$ is a linear combination of x and y, specifically $1x + 1y = x + y$ where $1 \in \mathbb{F}$ is the field's multiplicative identity. Hence W is closed under vector addition. By Theorem 1.3, W is a subspace of V. □

Problem 3. *Let*

$$M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Prove the span of $\{M_1, M_2, M_3\}$ is the subspace of $M_{2\times 2}(R)$ consisting of all symmetric 2×2 matrices.

Proof. By definition, a 2×2 matrix is of the form

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. This is also the linear combination $-a_1M_1 + a_2M_2 - a_3M_3$, so the set of all symmetric 2×2 matrices is a subset of the span of $\{M_1, M_2, M_3\}$. The latter is also a subset of the former, since for all $a_1, a_2, a_3 \in \mathbb{R}$ the linear combination $a_1M_1 + a_2M_2 + a_3M_3$ is the symmetric matrix

$$\begin{pmatrix} -a_1 & a_2 \\ a_2 & -a_3 \end{pmatrix}$$

Hence the two sets are exactly equal. Call it W; we will prove it is a subspace of $M_{2\times 2}(R)$. First notice that the the zero vector $\mathbf{0} \in M_{2\times 2}(R)$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is a symmetric matrix, so W contains it. Next, the sum of two symmetric matrices is also symmetric, so W is closed under addition:

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_2 + b_2 & a_3 + b_3 \end{pmatrix}$$

Finally, the scaling of a symmetric matrix is also symmetric, so W is closed under scalar multiplication:

$$c\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} = \begin{pmatrix} ca_1 & ca_2 \\ ca_2 & ca_3 \end{pmatrix}$$

By Theorem 1.3, W is a subspace of $M_{2\times 2}(R)$. □

Problem 4. *Determine if the set $\{1-3x^2+5x^3, 1-x^2+2x^3, 1+3x^2-4x^3\}$ is linearly independent or linearly dependent in $P_3(\mathbb{R})$. Justify your answer.*

Proof. We want to determine if there exists a non-zero linear combination of the three polynomials which gives 0, the zero vector in $P_3(\mathbb{R})$. Using Gaussian elimination where each column is a polynomial and the rows represent coefficients of the monomials $1, x^2, x^3$:

$$\begin{pmatrix} 1 & 1 & 1 \\ -3 & -1 & 3 \\ 5 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & -3 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us one possible tuple of coefficients $(2, -3, 1)$ among others. We can verify that $2(1-3x^2+5x^3) - 3(1 - x^2 + 2x^3) + 1(1+3x^2-4x^3) = 0$. By definition, the set is linearly dependent. □

Problem 5. *Recall the set of diagonal matrices in $M_{3\times 3}(\mathbb{R})$ is a subspace. Find a linearly independent set that spans this subspace (make sure to justify your answer!).*

Proof. One such set consists of the following matrices:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By definition, a 3×3 diagonal matrix is of the form

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. This is also the linear combination $a_1M_1 + a_2M_2 + a_3M_3$. Hence the span of $\{M_1, M_2, M_3\}$ is the set of diagonal matrices in $M_{3\times 3}(\mathbb{R})$. Next, notice that the zero vector in $M_{3\times 3}(\mathbb{R})$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is only equal to the linear combination $a_1M_1 + a_2M_2 + a_3M_3$ when $a_1, a_2, a_3 = 0$, the trivial solution. By definition, the set $\{M_1, M_2, M_3\}$ is linearly independent. □

Problem 6. *Let V be a vector space over a field $\mathbb{F}$ and suppose U and W are subspaces. Recall we can define the set*

$$U + W = \{x + y : x \in U, y \in W\}$$

Prove that $U + W$ is a subspace of V.

Proof. First, notice that the zero vector $\mathbf{0} \in V$ is in both U and W. Hence the sum $\mathbf{0} + \mathbf{0} = \mathbf{0}$ is also in $U + W$.

Next, for any two vectors $x_1 + y_1, x_2 + y_2 \in U + W$ where $x_1, x_2 \in U, y_1, y_2 \in W$, notice that $x_1 + x_2 \in U$ and $y_1 + y_2 \in W$, since both are subspaces. By definition, $(x_1 + x_2) + (y_1 + y_2) \in U + W$. By commutativity and associativity, this is equal to $(x_1 + y_1) + (x_2 + y_2)$, so $U + W$ is closed under vector addition.

Finally, for any scalar $c \in \mathbb{F}$ and vector $x + y \in U$ where $x \in U, y \in W$, notice that $cx \in U$ and $cy \in W$, since both are subspaces. By definition, $cx + cy \in U + W$. By distributivity of scalar multiplication over vector addition, this is equal to $c(x + y)$, so $U + W$ is closed under scalar multiplication. By Theorem 1.3, $U + W$ is a subspace of V. □