Math 115A Quiz 1

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Problem 1. *Honor Statement.*

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Problem 2. *Recall* $\mathbb{Z}/5 = \{0, 1, 2, 3, 4\}$ *with addition and multiplication defined "* mod 5*" (see the homework assignment if you need a refresher on this definition). Prove that all nonzero elements in* $\mathbb{Z}/5$ *have multiplicative inverses. That is, prove the following:*

For all nonzero $b \in \mathbb{Z}/5$*, there exists an element* $d \in \mathbb{Z}/5$ *such that* $b \cdot d = 1$.

Proof. Lifting to $\mathbb{Z}$, we want to find an integer x such that $b \cdot x = 1 + 5k$. Such an x, when reduced mod 5, would give an element $d \in \mathbb{Z}/5$ satisfying the multiplicative inverse property. Rearranging terms, we get

$$b \cdot x + 5k = 1$$

By the Euclidean algorithm, a solution exists when $\gcd(b, 5) = 1$. Since 5 is prime, this is always true. Hence for any nonzero element $b \in \mathbb{Z}/5$, there exists a corresponding multiplicative inverse. $\square$

Problem 3. *Let* $V = (a_1, a_2) : a_1, a_2 \in \mathbb{R}$*. Define addition and scalar multiplication on the set* V *by*

$$(a_1, a_2) + (b_1, b_2) = (a_1 b_1, a_2 + b_2), c \cdot (a_1, a_2) = (a_1, c a_2)$$

for $(a_1, a_2), (b_1, b_2) \in V$ *and* $c \in \mathbb{R}$*. Is* V *a vector space over* $\mathbb{R}$*? Justify your answer.*

Proof. One property of a vector space is that there exists a zero vector which acts as an additive identity. A zero vector (x_1, x_2) must satisfy

$$(a_1, a_2) + (x_1, x_2) = (a_1 x_1, a_2 + x_2) = (a_1, a_2)$$

The only solution is $x_1 = 1, x_2 = 0$. However, another requirement of a vector space is that each vector has an additive inverse. Suppose $(1, 0)$ is the zero vector. The inverse (y_1, y_2) of $(0, 0)$ must satisfy

$$(0, 0) + (y_1, y_2) = (0 \cdot y_1, 0 + y_2) = (1, 0)$$

This has no solution, since $0 \cdot y_1 = 0$ for all y_1. Hence a zero vector does not exist, and V is not a vector space over $\mathbb{R}$. $\square$

Problem 4. *Is the empty set* $\emptyset$ *a vector space over* $\mathbb{R}$*? Why or why not?*

Proof. The empty set is not a vector space over $\mathbb{R}$ (or any field) because it does not contain a zero vector. $\square$

Problem 5. *Let* $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = -a_3, a_2 = 4a_3\}$*. Prove that* W *is a subspace of the vector space* $\mathbb{R}^3$ *under the usual operations of addition and scalar multiplication on* $\mathbb{R}^3$.

Proof. First, notice that the zero vector $(0, 0, 0)$ is in W since $0 = -0$ and $0 = 4 \cdot 0$. Take any $c \in \mathbb{R}$ and two vectors $\mathbf{a}, \mathbf{b} \in W$ where

$$\mathbf{a} = (a_1, a_2, a_3) = (-a_3, 4a_3, a_3)$$

$$\mathbf{b} = (b_1, b_2, b_3) = (-b_3, 4b_3, b_3)$$

The sum $\mathbf{a} + \mathbf{b} = (-a_3 + (-b_3), 4a_3 + 4b_3, a_3 + b_3) = (-(a_3 + b_3), 4(a_3 + b_3), a_3 + b_3)$ is in W, so W is closed under addition. The scalar product $c \cdot \mathbf{a} = (c \cdot -a_3 . c \cdot 4a_3, c \cdot a_3) = (-(ca_3), 4(ca_3), ca_3)$ is in W, so W is closed under scalar multiplication. Hence by the theorem proved in class, W is a subspace. □

Problem 6. *Let V be a vector space over the field $\mathbb{F}$. Prove for all scalars $a, b \in F$ and vectors $x, y \in V$, $(a + b) \cdot (x + y) = ax + bx + ay + by$.*

Proof. By distributivity of scalar multiplication over vector addition,

$$(a + b) \cdot (x + y) = (a + b) \cdot x + (a + b) \cdot y$$

By distributivity of scalar multiplication over scalar addition,

$$(a + b) \cdot x + (a + b) \cdot y = ax + bx + ay + by$$

□