Problem 5. [5 pts]
Prove that following matrix A is diagonalizable

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 4 & 6 & 4 \\ -2 & -2 & 0 \end{pmatrix}$$

4

and give a formula for A^n for every $n \geq 1$, as a product of 3 explicit matrices.

$$\det\begin{pmatrix} 4-t & 2 & 2 \\ 4 & 6-t & 4 \\ -2 & -2 & -t \end{pmatrix} = (4-t)[t^2-6t+8] - 2[-4t+8] + 2[-8+12-2t]$$

$$= 4t^2 - 24t + \cancel{32} - t^3 + 6t^2 \cancel{-8t}$$

$$\cancel{+8t} \cancel{-16} -16 + 24 - 4t$$

$$= -t^3 + 10t^2 - 28t + 24$$

$$= -1(t^3 - 10t^2 + 28t - 24)$$

$$\begin{array}{r} t^2 - 8t + 12 \\ t-2 \overline{) t^3 - 10t^2 + 28t - 24} \\ t^3 - 2t^2 \\ \hline -8t^2 + 28t \\ -8t^2 + 16t \\ \hline 12t - 24 \end{array}$$

$8 - 40 + 56 - 24$

$(t-6)(t-2)$

$\lambda_1 = 2 \quad \lambda_2 = 6$

$a-$ $\quad a+$ $\quad a+b$

$E\lambda_1$:

$$\begin{pmatrix} 2 & 2 & 2 \\ 4 & 4 & 4 \\ -2 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} a+b+c=0 \\ c=a+b \end{matrix}$$

$$E_{\lambda_1} = \{(a, b, -a-b) \mid a, b \in \mathbb{R}\}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$E\lambda_2$:

$$\begin{pmatrix} -2 & 2 & 2 \\ 4 & 0 & 4 \\ -2 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} \cancel{0} & \cancel{1} & \cancel{2} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{\lambda_2} = \{(-a, -2a, a) \mid a \in \mathbb{R}\}$$

$$\Rightarrow \left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$Q = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

$$D = Q^{-1}AQ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$A = QDQ^{-1}$$

$$A^n = Q^n \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 6^n \end{pmatrix} Q^{-1\,n}$$

What is Q^{-1} explicitly

Problem 1. [5 pts]
Answer only by "True" (T) or "False" (F) **without justification.**

(a) If $\{W_i\}_{i\in I}$ is a collection of subspaces of a vector space V, then their intersection $\bigcap_{i\in I} W_i$ is a subspace of V.

T ✓

(b) If $T\colon V \to W$ is linear and both its kernel (nullspace) and its image (range) have finite dimension then V has finite dimension as well.

T ✓

(c) If $T_1\colon V \xrightarrow{\cong} W$ and $T_2\colon V \xrightarrow{\cong} W$ are isomorphisms then $T_1 + T_2$ is an isomorphism as well.

F ex: I_V and I_V ✓

for $T_1: V \to V$
$T_2: V \to V$

(d) Over the field of complex numbers, all square matrices are diagonalizable.

F ex: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ✓

(e) The matrix $\begin{pmatrix} 2 & 0 & a \\ 0 & 2 & b \\ 0 & 0 & 3 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$ is diagonalizable for all $a, b \in \mathbb{R}$.

T ✓

5

Problem 2. [5 pts]
Let V be a finite-dimensional vector space and $T\colon V \to V$ linear such that rank(T) = rank(T^2). (Recall that Ker$(T) = N(T)$ and Im$(T) = R(T)$, and that $W_1 \oplus W_2$ means $W_1 + W_2$ together with $W_1 \cap W_2 = \{0\}$.)

(a) Show that Im(T) = Im(T^2) and Ker(T) = Ker(T^2).

(b) Show that $V = \text{Im}(T) \oplus \text{Ker}(T)$.

a) $\text{Im}(T^2) \subseteq \text{Im}(T)$

same dim

As $\text{Im}(T^2) = \text{Im}(T)$

$w \in \text{Im}\, T^2$

$w = T^2 v$

$w = T(T(v))$

$\in \text{Im}(T)$

$\text{Ker}(T) \subseteq \text{Ker}(T^2)$

$\Rightarrow \text{Ker}(T) = \text{Ker}(T^2)$

b) If $v \in \ker T \cap \text{Im}\, T$

$\exists\ w \in V \quad v = Tw$

$T w^2 = Tv = 0$

$Tw = v = 0$

$\ker(T) = \ker(T^2)$

$T^2 w = 0$

$Tw = 0$

$\Rightarrow \ker T \cap \text{Im}\, T = \{0\}$

$\dim(\ker T) + \dim \text{im}\, T = \dim V$

0/5

Math 115A, Lect. 3
Midterm 2
2019 May 22

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Signature: Shangyu Hsu

– NO BOOK – NO CALCULATOR –

You can use both sides of the sheet for each problem. If you use extra sheets, write your name on them, and indicate which problem you are treating.

All answers must be stated clearly.

All answers must be JUSTIFIED (except for Problem 1).

Good luck!

– SCORE –

1. 5 2. 0

3. 1 4. 2

5. 4

Total 12

Problem 3. [5 pts]
Is the following claim true or false: "*Let V be a finite-dimensional vector space and let $T, S\colon V \to V$ be two linear transformations. If T and S are diagonalizable then $T+S$ is diagonalizable*".

Let $A = [T]_\beta$ & $B = [S]_\beta$ for β = basis of V.

$\exists$ basis β_T consisting of eigenvectors w.r.t. eigenvalues of T
basis B_S consisting of eigenvectors w.r.t. eigenvalues of S

$\boxed{\text{false.}}$ ✓ 1

ex:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{array}{c} \mathbb{R}^2 \to \mathbb{R}^2 \\ x \mapsto Ax \end{array}$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Problem 4. ~~[5 pts]~~ -1

Give an **explicit example** of a linear transformation $T: V \to V$ with two eigenvalues: $\lambda_1 = 7$ with algebraic multiplicity 3 and geometric multiplicity 2, and $\lambda_2 = -4$ with algebraic multiplicity 4 and geometric multiplicity 1.

$\dim(A - 7I) = 2$

5 $\dim(A + 4I) = 1$ let β = basis of V s.t.

$V = ?$

$$A = \begin{bmatrix} 7 & & & & & & \\ & 7 & & & & & \\ & 1 & 7 & & & & \\ & & & -4 & 1 & & \\ & & & & -4 & 1 & \\ & & & & & -4 & 1 \\ & & & & & & -4 \end{bmatrix} = [T]_\beta$$

justify?

2/5

$T: V \to V$

$T(v) = \phi_\beta^{-1}(A\,\phi_\beta(v))$ $v \in V$

$$\begin{pmatrix} -4 & 1 & 0 & 0 & & & \\ 0 & -4 & 1 & 0 & & 0 & \\ 0 & 0 & -4 & 1 & & & \\ 0 & 0 & 0 & -4 & & & \\ & & & & 7 & 1 & 0 \\ & 0 & & & 0 & 7 & 1 \\ & & & & 0 & 0 & 7 \end{pmatrix}$$

* jordan block

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ X \\ 0 \\ Y \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ A \\ 0 \\ 0 \\ 0 \end{pmatrix}$$