$(a_1, 0 \ldots)$
$(0, \ldots)$

Problem 5. [5 pts]
Let $V = \{(x_i)_{i\geq 1} \mid x_i \in F\}$ be the F-vector space of sequences in F. Find a linear transformation $T\colon V \to V$ whose kernel (nullspace) is zero but whose image (range) is not the whole of V.

$$\boxed{T(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)}$$ for all ... $\in F$

For $(0, a_1, a_2, \ldots) = (0, 0, \ldots)$,
$a_1 = a_2 = \cdots = 0$ must be true.
$\rightarrow \ker(T) = (0, 0, 0 \ldots)$ ✓

$\underbrace{\mathrm{Im}(T)}_{\text{space}} = \{\underbrace{(0, a_1, a_2, \ldots)}_{\text{one vector}} \mid$ where $a_1, a_2 \ldots \in \mathbb{F}.\}$

Let $X = (a', a_1, a_2, \ldots)$ where $a' \in F$ and $a' \neq 0$

$X \in V$ since $a' \in F$ but $X \notin \mathrm{Im}(T)$ since $a' \neq 0$.
✓

4/5.

Problem 2. [5 pts]

Let W_1, W_2 be two subspaces of a finite-dimensional vector space V. Recall that $W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2\}$ is a subspace of V. (No need to reprove this.) What is $\dim(W_1 + W_2)$?

Thm: $B = \{u_1, \dots, u_n\} \subset V$ is a basis of V iff $v \in V$ is uniquely written as $v = \sum_1^n a_i u_i$.

Let $x \in W_1 + W_2$. Then $x = w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2$.

Let W_1 have basis $\{u_1, \dots, u_n, v_1, \dots v_k\}$ what is

Let W_2 have basis $\{y_1, \dots, y_m, v_1, \dots, v_k\}$ this??

So $W_1 \cap W_2 = \{v_1, \dots, v_k\}$.

def. of a basis, $w_1 = \sum_{i=1}^n a_i u_i + \sum_{i=1}^k b_i v_i$ & $w_2 = \sum_{i=1}^m c_i y_i + \sum_{i=1}^k d_i v_i$ uniquely,

where $a_i, b_i, c_i, d_i \in F$ for $1 \le i \le n$, $1 \le i \le k$, & $1 \le i \le m$.

Then x can be written uniquely as

$$x = \sum_{i=1}^n a_i u_i + \sum_{i=1}^k (b_i + d_i) v_i + \sum_{i=1}^m c_i y_i.$$

So $\{u_1, \dots u_n, v_1, \dots, v_k, y_1, \dots, y_m\}$ is a basis of $W_1 + W_2$ by the theorem.

$$\dim(W_1 + W_2) = m + n + k = m + k + n + k - k$$

$$= \boxed{\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)}$$

Math 115A, Lec. 4
Midterm 1
2017 October 20

Name: Peng, Melissa
Please put your last name first and print clearly

UID: 104 – 611 – 280

Signature: M Pg

– NO BOOK – NO CALCULATOR –

You can use both sides of the sheet for each problem. If you use extra sheets, write your name on them, and indicate which problem you are treating.

All answers must be stated clearly.

All answers must be justified (except for Problem 1).

Verbiage will be ignored or counted negatively if wrong. Good luck!

– SCORE –

1. 3 2. 2

3. 4 4. 5

5. 4

Total 18

Problem 1. [5 pts]
Answer only by "True" or "False" **without justification.**

?. (a) There exists a field $\mathbb{F}$ in which $1+1+1=0$.

F

(b) Every linearly independent subset L of a vector space V is the basis of a subspace of V.

F

(c) In a vector space V of dimension n, every basis of V has n elements.

T ✓

(d) If $T\colon V \to W$ is linear and $v_1, \ldots, v_n$ is a basis of V then $T(v_1), \ldots, T(v_n)$ is a basis of W.

F ✓

odd dim

(e) There exists a linear transformation $T : \mathbb{R}^7 \to \mathbb{R}^7$ such that $\mathrm{Im}(T) =$ $\mathrm{Ker}(T)$ (that is, $R(T) = N(T)$).

F

✓

3/5

$Im(T) \subset V$
$V \subset Im(T)$

Problem 4. [5 pts]
Let V be a finite-dimensional vector space and let $T: V \to V$ be a linear transformation from V to itself. Suppose that the kernel (nullspace) of T is zero. Prove that the image (range) of T is the whole of V.

WTS: $T: V \to V$ is onto. ✓

① Thm: Let $\# V = \# W$ & $T: V \to W$ be linear. Then (a-d) are equivalent.
a) T is one to one
b) T is onto
c) $RK(T) = \dim(W)$ ✓

② Rank-nullity thm: Let V be fin. dim. & $T: V \to W$ be linear. $\dim(V) = rk(T) + \dim(\ker(T))$

Since $\ker(T) = 0$, $\dim(\ker(T)) = 0$.
By Thm ②, $\dim(V) = rk(T) + 0$
$\dim(V) = rk(T)$
$\# V = \# V$ obviously.
By Thm ①, since c) is true then a) is true and T is onto.

overkill! (one thm is enough)

$\therefore Im(T) = V$. ✓

✓

5/5.

2) B is LI

$a_1(1+x+x^2) + a_2(2+x+x^2) + a_3x^2 + a_4x^3 = 0$

$(a_1+2a_2) + (a_1+a_2)x + (a_1+a_2+a_3)x^2 + a_4x^3 = 0$

$\underset{0}{\shortparallel} \quad \underset{0}{\shortparallel} \quad \underset{0}{\shortparallel} \quad \underset{0}{\shortparallel}$

Only the trivial soln, $a_1 = a_2 = a_3 = a_4 = 0$ makes all the coeffs 0.

→ B is LI

Since B is LI & span(B) = V, B is a basis of V.

vague
(proof not ok)

2/3.

Problem 3. [5 pts]
Consider $V = P_3(\mathbb{R})$ the real vector space of polynomials of degree at most 3.

(a) Show that $v_1 = 1+X+X^2$ and $v_2 = 2+X+X^2$ are linearly independent.

(b) Complete $\{v_1, v_2\}$ into a basis of V. $1, x, x^2, x^3$

a) Let $a_1(1+x+x^2) + a_2(2+x+x^2) = 0$? ?

$(a_1 + 2a_2) + (a_1+a_2)x + (a_1+a_2)x^2 = 0$

$$\underbrace{(a_1+2a_2)}_{=0} \quad \underbrace{(a_1+a_2)}_{=0} \quad \underbrace{(a_1+a_2)}_{=0}$$

$$\begin{aligned} a_1 + a_2 &= 0 \\ -)\ a_1 + 2a_2 &= 0 \\ \hline -a_2 &= 0 \\ a_2 &= 0 \end{aligned}$$

$a_1 = -a_2 \rightarrow a_1 = 0$

Since the only solution is $a_1 = a_2 = 0$, $\{v_1, v_2\}$ is lin. indep. $\square$ ✓

2/2.

b) Let $v \in V$. Then $v = a + bx + cx^2 + dx^3$, where $a, b, c, d \in \mathbb{R}$.

Claim: $B = \boxed{\{1+x+x^2,\ 2+x+x^2,\ x^2, x^3\}}$ is a basis. ✓

1) $\operatorname{span}(B) = V$.

$a_1(1+x+x^2) + a_2(2+x+x^2) + a_3x^2 + a_4x^3 = a + bx + cx^2 + dx^3$, where $a_1, a_2, a_3, a_4 \in$

$(a_1 + 2a_2) + (a_1+a_2)x + (a_1+a_2+a_3)x^2 + a_4x^3 = a + bx + cx^2 + dx^3$

$d = a_4,\ c = a_1 + a_2 + a_3,\ b = a_1 + a_2,\ a = a_1 + 2a_2$

$\rightarrow v \in \operatorname{span}(B)$ and $v \subset \operatorname{span}(B)$.

quantifiers! the a, b, c, d are given not $a_1..a_4$ are not all

Let $b \subset \operatorname{span}(B)$. Then $b = a_1(1+x+x^2) + a_2(2+x+x^2) + a_3(x^2) + a_4x^3$.

$b = (a_1 + 2a_2) + (a_1+a_2)x + (a_1+a_2+a_3)x^2 + a_4x^3$

Since $a_1, a_2, a_3, a_4 \in \mathbb{R}$ & $\mathbb{R}$ is closed under addition & scalar mult, the coeffs. of b are all $\in \mathbb{R}$.

$\rightarrow b \in V$ and $\operatorname{span}(B) \subset V$.

Since $\operatorname{span}(B) \subset V$ & $V \subset \operatorname{span}(B)$, $\operatorname{span}(B) = V$. Cont'd on back