1. (4 points) 1. Let V be a vector space of dimension n. For each $0 \leq r \leq n$, show that there is a subspace of V whose dimension is r.

2. Let $v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$. Is the matrix $v = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ a linear combination of v_1, v_2 and v_3?

① if $\dim(W) = n$, $W = V$ ✓ (both are subspaces)
if $\dim(W) = 0$, $W = \{\vec{0}\}$ ✓

if $\dim(W) \in \{1, \ldots, n-1\}$, can take a subset of size r of the basis of V, There are n vectors in the basis of V, $r < n$ so there are r.
The span of this subset is a linear combination, closed under scalar mult, addition & contains 0. This is contained in V because these vectors come from V.

$\exists$ subspace $W \subset V \mid \dim(W) = r$ ☒

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② No it is not.

The only v_i vectors that have A_{12} and A_{21} nonempty is v_3. This means that

v_{12} and v_{21} need to be equal in magnitude but of opposite sign.

2. (4 points) Which of the following subsets of $V = M_{n \times n}(F)$ are subspaces?

   1. The subset of invertible matrices.
   2. The subset of upper triangular matrices. Here an upper triangular matrix is one for which all the entries below the diagonal are zero.

For each of the above subsets that is a subspace, find a basis and its dimension.

① the zero matrix is <u>NOT</u> invertible

$\Rightarrow$ invertible matrices not a ✓

subspace

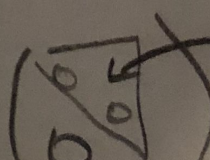
② The set of upper triangular is a subspace.

$0 \in$ set ✓

upper tri + upper tri = upper tri. ✓ ✓

scalar • upper tri = upper tri. ✓

$B = \{ w \in V \mid$ one value in V is 1 above/= to the diagonal is 1, with all other entries being 0 $\}$



1 1 in here

$\boxed{\text{Dim}(W) = \sum_{i=1}^{n} i}$ ✓

4

3. (4 points) Let V be a finite dimensional vector space and let W_1 be a subspace of V. Show that there exists another subspace W_2 of V such that

1. $V = W_1 + W_2$, and
2. $W_1 \cap W_2 = 0$.

$|B| = n$ $|L| = m$

Let B be a basis for V (generates) V. Let L be a basis for W_1 (linearly independent set).

$\exists\ H \subseteq B \mid L \cup H$ generates V. $|H| = n-m$

(replacement)

$W_2 = H$, → $W = span(H)$

Therefore there is some W_2 that when added to W_1 will generate V (and nothing more because the vectors of W_2 come from the basis of V).

Any set that generates V must have a basis of size n at the very least.

$L \cup H$ has at most $n - m + m = n$ vectors. Since need $\geq n$ vectors, none of the vectors in L or H can be cancelled out (taken out of the set due to redundancy).

Therefore $L \cap H = \emptyset$?

$(W_1 \cap W_2 = 0)$

3

This meets both criteria $\boxtimes$

4. (4 points) Let V be a vector space over a field F and let $W \subset V$ be a subspace. Consider the set

$$E = \{T : V \to V \mid T \text{ is a linear transformation such that } T(W) \subset W\}.$$

For $T_1, T_2 \in E$ and $c \in F$, define

1. $(T_1 + T_2)(v) = T_1(v) + T_2(v)$,
2. $(cT_1)(v) = cT_1(v)$.

Show that these operations make E into a vector space.

4

E: Cld under add/scalar mult?

1) $(T_1 + T_2)(v) = T_1(v) + T_2(v) = T_2(v) + T_1(v) = (T_2 + T_1)(v)$ ✓

2) $((T_1 + T_2) + T_3)(v) = (T_1 + T_2)(v) + T_3(v) = T_1(v) + T_2(v) + T_3(v)$
$= T_1(v) + (T_2 + T_3)(v) = (T_1 + (T_2 + T_3))(v)$ ✓

3) $(1 \cdot T_1)(v) = 1 \cdot T_1(v) = T_1(v)$ ✓

3) $\exists 0$ $(T_1 + 0_T)(v) = T_1(v) + 0_T(v) = T_1(v)$ ✓

0_T function that sends vector to 0 vector in V (which exists bc V is vsp)

4) $((ab)T_1)(v) = (ab)T_1(v) = a(bT_1(v)) = a[(bT_1)(v)]$ ✓

5) Inverse: $(T_1 + T_1')(v) = T_1(v) + T_1'(v) = 0$ ✓

$v + v' = 0$

(There is an inverse function that returns the inverse of the output (which exists because V is a vector space))

6) $[(a+b)T_1(v)] = (a+b)T_1(v) = aT_1(v) + bT_1(v)$ ✓

7) $[a(T_1 + T_2)](v) = a[T_1 + T_2](v) = a[T_1(v) + T_2(v)] = aT_1(v) + aT_2(v)$ ✓

These all work because $T(v)$ is a vector $\in V$ (which is a vector space)

E is a vsp because the criteria are satisfied $\boxed{}$

5. (4 points) Let V be the vector space over $\mathbb{R}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that the set

$$S = \{\sin^n(x) \mid n \geq 0\}$$

is linearly independent in V.

4

Show not linearly dependent:

Show no non trivial relation $\left(c_0 1 + c_1 \sin(x) + c_2 \sin^2(x) + \dots + c_n \sin^n(x) = 0 \iff c_i = 0 \right)$

$\sin^0(x) = 1$ (lowest term 1)

$\sin^1(x) = x - \frac{x^3}{3!} + \dots$ (lowest term x)

$\sin^2(x) = \left(x - \frac{x^3}{3!} \dots\right)^2$ (lowest term x^2)

$n=0$ is the only term with x^0 term (nothing to cancel it out, like $-x^0$), this means that $c_0 = 0$. $n=1$ is the only term with x^1 so it needs to be $c_1 = 0$ for the same reason. $n=2$ only one with $x^2 \rightarrow c_2 = 0$.

Next $n=3$ is NOT the only one with x^3 term but since all previous x^3 terms were 0'd because of single terms, there is nothing to cancel it out $\rightarrow c_3 = 0$.

This chain (induction) works its way all the way to $n=n$.

Since there is no non-trivial relation, S is linearly independent ☒