Math 115A – Midterm 2

There are four problems c

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Problem 1:

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a: (2 points) Suppose that $\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$, and that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T(v_1) = 3v_1 - v_2$, and with $T(v_2) = 2v_1$. What is the matrix for T, with respect to the basis $\{v_1, v_2\}$?

$$\hookrightarrow A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

0

b: (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T([x, y]) = [-y, x]$. Consider the basis for $\mathbb{R}^2$ given by $\{[2, 1], [1, 2]\}$. What is the matrix for T, with respect to this basis?

$$T([2,1]) = [-1,2] =$$
$$T([1,2]) = [-2,1] =$$

0

c: (4 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by reflection through the $x = z$ plane. Find a basis $\{v_1, v_2, v_3\}$ such that the matrix for T with respect to this basis is a diagonal matrix.

$$T(v_1) = \alpha v_1 + 0 + 0$$
$$T(v_2) = 0 + \beta v_2 + 0$$
$$T(v_3) = 0 + 0 + \gamma v_3$$

$$\hookrightarrow \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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Problem 2: Suppose that A is a 4×4 matrix with real entries, and that the first column is a linear combination of the other three. Use the properties of determinants (given on the last page of this exam) to show that this matrix has determinant 0.

$$\det A = \sum_i (-1)^{i+1} a_i \overline{A_i} \quad \text{minor matrix}$$

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

$$v_1 = \alpha v_2 + \beta v_3 + \gamma v_4$$

$$\det A = \det \left[\alpha v_2 + \beta v_3 + \gamma v_4 \,|\, v_2 \,|\, v_3 \,|\, v_4 \right]$$

(Property 4)

$$= \det \left[\alpha v_2 | v_2 | v_3 | v_4 \right] + \det \left[\beta v_3 + \gamma v_4 | v_2 | v_3 | v_4 \right]$$

$$= \det \left[\alpha v_2 | v_2 | v_3 | v_4 \right] + \det \left[\beta v_3 | v_2 | v_3 | v_4 \right] + \det \left[\gamma v_4 | v_2 | v_3 \ldots \right.$$

$$= \alpha \det \left[v_2 | v_2 | v_3 | v_4 \right] + \beta \det \left[v_3 | v_2 | v_3 | v_4 \right] + \gamma \det \left[v_4 | v_2 \ldots \right.$$

$$= \alpha (0) + \beta \det \left[v_2 | v_3 | v_3 | v_4 \right] - \gamma \det \left[v_2 | v_4 \ldots \right.$$

(swap property also)

$$= 0 - \beta (0) + \gamma \det \left[v_2 | v_3 | v_4 | v_4 \right]$$

(swap)

$$= 0 - 0 + \gamma (0)$$

$$= 0$$

how $\rightarrow$ $\det [x | x | y_i \ldots]$

where $y_i \neq y_j$ and $y_i \neq x$ for any given i, j

then by (prop 3), switch col 1 & 2

$$\det [x | x | y_i \ldots] = -\det [x | x | y_i \ldots]$$

so $2 \det [x | x | y_i \ldots] = 0$

$$\Rightarrow \det [x | x | y_i \ldots] = 0$$

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Problem 3: Let V be a finite dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation represented by a matrix A. Suppose that λ is an eigenvalue for T. Show that λ is a root of the characteristic polynomial $\chi(t) = \text{Det}(A - tI)$, where I is the identity matrix.

$\chi(t) = \text{Det}(A - tI)$. show $\chi(\lambda) = 0$

for an e-vec v, for λ

$Av = \lambda v$

$Av = \lambda I v$

$(A - \lambda I)v = 0$

what if $v = 0$?

-2

8 Conclusion?

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Problem 4: Let V be an inner product space over $\mathbb{R}$, with inner product $< \_, \_ >$. Let U be a three-dimensional subspace of V, with an orthonormal basis $\{u_1, u_2, u_3\}$. The orthogonal projection onto U is the linear transformation $T : V \to V$ defined by $T(v) = < u_1, v > u_1 + < u_2, v > u_2 + < u_3, v > u_3$. If a vector w is in the kernel of this transformation, show that w is orthogonal to every vector in U.

$T(w) = 0 = \langle u_1, w\rangle u_1 + \langle u_2, w\rangle u_2 + \langle u_3, w\rangle u_3$

u_1, u_2, u_3 are linearly in dep.

So the scalars $\langle u_1, w\rangle = 0$

$\langle u_2, w\rangle = 0$

$\langle u_3, w\rangle = 0$

So w is orthogonal to each of the basis vectors of U.

Any vector $x \in U$ can be represented as linear combo of u_1, u_2, u_3

so $\langle x, w\rangle = \langle \alpha u_1, w\rangle + \langle \beta u_2, w\rangle + \langle \gamma u_3, w\rangle$

$= \alpha \langle u_1, w\rangle + \beta \langle u_2, w\rangle + \gamma \langle u_3, w\rangle$

$= \alpha 0 + \beta 0 + \gamma 0$

$= 0$

So w is orthogonal to all vectors in U.

Properties of determinants:

1) The determinant of the identity matrix is 1.

2) If A is a square matrix, and B is the matrix obtained from A after multiplying one column by a real number k, then $\text{Det}(B) = k\text{Det}(A)$.

3) If A is a square matrix, and B is the matrix obtained from A by switching two columns, then $\text{Det}(B) = -\text{Det}(A)$.

4) Suppose that A, B and C are square matrices, the same except for their ith columns. If the ith column of A is v, the ith column of B is v', and the ith column of C is $v + v'$, then $\text{Det}(C) = \text{Det}(A) + \text{Det}(B)$.