Math 115A – Midterm 1

There are four problem

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Problem 1: Let V be a vector space over a field F, and let U_1 and U_2 be subspaces of V. Let W be the set $W = U_1 \cap U_2$ (the intersection of U_1 and U_2), meaning the set of all vectors which are in both U_1 and U_2. Prove that W is a subspace of V.

- $0 \in U_1$ and $0 \in U_2$ (they are subspaces)
  so $0 \in U_1 \cap U_2$

- Let $x, y \in U_1 \cap U_2$
  $x + y \in U_1$, because U_1, a subspace, is closed under vector addition
  $x + y \in U_2$ because U_2 " " " "
  so for arbitrary $x, y \in U_1 \cap U_2$, $x + y \in U_1 \cap U_2$
  so $U_1 \cap U_2$ is closed under vector addition

- let $k \in F$
  $kx \in U_1$ because U_1, a subspace, is closed under scalar multiplication
  $kx \in U_2$ because U_2 is also a subspace and closed under scalar multiplicaon
  so $kx \in U_1 \cap U_2$
  so $U_1 \cap U_2$ is closed under scalar multitplication

so $U_1 \cap U_2$ (W) is a subspace of V

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Problem 2: Prove that any composition of two linear transformations is also a linear transformation.

Let V, W, U vector spaces over a field F,
let $T: V \to W$, $S: W \to U$, T&S are linear transformation
Let $x, y \in V$

Consider $S(T(\vec{v}))$ or $S \circ T : V \to U$

We'll need to show $S \circ T(x+y) = S \circ T(x) + S \circ T(y)$; $k S \circ T(x) = S \circ T(kx)$

$S \circ T(x+y) = S(T(x+y))$

$= S(T(x) + T(y))$; T is a linear transformation on its own and preserves vector addition

$= S(T(x)) + S(T(y))$; $T(x), T(y) \in W$, and S is a linear transformation on its own

$= S \circ T(x) + S \circ T(y)$ ✓

$S \circ T(kx) = S(T(kx))$

$= S(k[T(x)])$; $T(x)$ is a linear transformation and preserves scalar multiplication

$= k S(T(x))$; $S(x)$ " " " "

$= k(S \circ T(x))$ ✓

$S \circ T$ preserves vector addition and scalar multiplication, so $S \circ T$ is a linear transformation

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spanning a space doesn't imply basis!

Problem 3: Suppose that V and W are both vector spaces over a field F, and that $T : V \to W$ is a linear transformation. Suppose that $k_1, \dots k_n$ are vectors in the kernel of T, and that they span the kernel. Suppose that $r_1, \dots r_m$ are vectors in V such that the vectors $T(r_1), \dots T(r_m)$ span the image of T. Show (using only the definitions) that any vector in V can be expressed as a linear combination of the vectors $k_1, \dots k_n, r_1, \dots r_m$.

prove without rank-nullity!

We are given $\mathrm{Rank}(T) \cancel{=} m$; $\mathrm{nullity}(T) \cancel{=} n$

we know, by Rank-~~nullity~~ Thm, that then,

$\mathrm{Dim}(V) = \cancel{m+n}$

so, then we need to ~~start~~ a basis of $m+n$ vectors here!

So $k_1 \dots k_n$ are ~~linearly indep.~~

$r_1 \dots r_m$ are ~~linearly indep.~~

If we assume their union is not, then

$$\sum_{i=1}^{n} \alpha_i k_i + \sum_{j=1}^{m} \beta_j r_j = 0$$

+1

then $$\sum_{i=1}^{n} \alpha_i k_i = \sum_{j=1}^{m} \beta_j r_j$$

so then any vector in the kernel is also in the image

why?

now, let $v \in V \neq 0$

$$T(v) = T\left(\sum_{i=1}^{n} \alpha_i k_i + \sum_{j=1}^{m} \beta_j r_j\right)$$

$$= 0 + T\left(\sum_{j=1}^{m} \beta_j r_j\right) \in \mathrm{Im}(T)$$

So v is a valid member of domain V

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Problem 4: Let V_n be an n-dimensional vector space over a field F. Suppose that, for each integer $i \in \{0, 1, \ldots n\}$, there is an i-dimensional subspace V_i of V, and that these spaces are all nested, so $V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n$. Show that there is a basis $b_1, \ldots b_n$ for V, such that for each $i \in \{1, \ldots n\}$, the vector b_i is in V_i but not in V_{i-1}.

which one?

base case: $i = 1$

We'll take the basis vector $b_1 \in V_1$ and use that.

let's assume that for V_n, we have basis vectors $b_1 \ldots b_n$ so that each b_i is in V_i but not V_{i-1} (subspaces like the above)

Then for V_{n+1}, we can take a vector $v \in V_{n+1}$ but not $v \notin V_n$, (without loss of generality)

we'll show, then, that $\{b_1, \ldots b_n, v\}$ is a basis:

they are linearly independent:

assuming they're not, then $\sum_{i=1}^{n} \alpha_i b_i + \alpha_{n+1} v = 0$

then $\sum_{i=1}^{n} \alpha_i b_i = -\alpha_{n+1} v$ for α's $\in F$ and not all 0

now, if $\alpha = 0$, then we have that the b_i vectors are linearly dependent Σ

if $\alpha \neq 0$, then $-\frac{1}{\alpha_{n+1}} \sum_{i=1}^{n} \alpha_i b_i = v$ and

we have that v can be written as a linear comb of the b_i vectors, $\Rightarrow v \in V_n$ Σ
$(v \in \text{span}(b_i))$

so $\{b_1 \ldots b_n, v\}$ is linearly independent

they span V_{n+1}

let's talk about $x \in V$, x

Then w/o loss of generality let's say

$x = \sum_{i=1}^{n} \alpha_i b_i + \vec{u}$ for α's $\in F$ and $u \in V$, $u \neq 0$ and $u \notin \text{span}\{b_1 \ldots b_n\}$

so, if $u = 0$, then $x \in \text{span}\{b_1, \ldots b_n, v\}$

otherwise,

$x - \sum_{i=1}^{n} \alpha_i b_i = u$, then $u \in V_{n+1}$ but $u \notin V_n \Rightarrow$ we have v, which can be scaled $\alpha_i v$ st $\alpha_{n+1} v = u$

this way, we can write an arbitrary vector $x \in V_{n+1}$ as a linear combination of $\{b_1 \ldots b_n, v\}$, so $x \in \mathrm{span}\{b_1 \ldots b_n, v\}$

so then $\{b_1 \ldots b_n, v\}$ is a basis for V_{n+1}

$\square$