1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) In the vector space $\mathbb{C}[x]$, the set $\{ x^2 - x, x^2 + 1 \}$ is

   A. linearly dependent
   B. linearly independent
   C. a spanning set
   D. none of the above

$x(x-1)$ $(x+1)(x-1)$

$\lambda(x^2 - x) + \mu(x^2+1)$

$(\lambda+\mu)x^2 - \lambda x + \mu = 0.$

The following two questions concern the subsets

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a^2 + b^2 = c \right\} \subseteq \mathbb{R}^3$$

$$V = \{ p(x) \in \mathbb{R}_2[x] \mid p(1) = \lambda \} \subseteq \mathbb{R}_2[x]$$

if $\lambda \neq 0$

$p(x), q(x) \in V.$

$[p(x)+q(x)](1) = p(1) + q(1)$

$\Rightarrow \lambda \neq \lambda.$

for some choice of $\lambda \in \mathbb{R}$. Recall that $\mathbb{R}_2[x]$ is the space of polynomials of degree at most 2.

   (b) (1 point) Which of the following is a true statement?

   A. Both U and V are subspaces regardless of the value of $\lambda \in \mathbb{R}$.
   B. Only U is a subspace.
   C. V is a subspace for any λ.
   D. Only V is a subspace when $\lambda = 0$.

$ax + b = \lambda$ $a+b=0.$ $2a+b$

$(a_1+a_2)^2 + (b_1+b_2)^2.$ $p = ax+b.$

$a+b=0.$

$2a+b=0.$

$p(1) = \lambda. \Rightarrow a+b = \lambda.$

$p(2) = 2a+b = 2\lambda$

$ax+b=\lambda$ $a+b=\lambda$ a_1 a_2

b_1 b_2

$a(kx)+b = ka+b.$

$a_1^2+b_1^2$ $a_2^2+b_2^2$

$a(x+1)+b = 2a+b. = 0$

$a+b=\lambda.$

   (c) (1 point) When $\lambda = 0$, the subspace V has dimension

   A. 1
   B. 2
   C. 3
   D. 4

$\{1, x, x^2\}$

$a+b=0.$

$-a = b+c.$

$ax - a.$ $(x-1)$

$p = -b-c+bx+cx^2$

$= b(x-1) + c(x^2-1)$

$\dim = 2$

(d) (1 point) Let U and W be two, finite dimensional subspaces of a vector space V. Which of the following statements is true?

A. We must have $U \cap W = \{0\}$.

B. If $U \cap W = \{0\}$ then $U + W = V$. $u+w = x+y,\ x \in u,\ y \in V$.

C. $\dim U + \dim W \geq \dim(U + W)$.

D. The dimension of $U + W$ is unrelated to $\dim U$ and $\dim W$.

$\dim U + \dim W = \dim U \cup W + \dim U \cap W$.

select a basis $\alpha = \{v_1 \cdots v_k\}$ of $U \cap W$.

expand α to basis of U $\{v_1 \cdots v_k, u_1 \cdots u_m\} = \beta$

V $\{v_1 \cdots v_k, w_1 \cdots w_n\} = \gamma$.

$\dim U = k+m$. $\dim V = k+n$.

$B \cup \gamma = \{v_1 \ldots v_k, u_1 \ldots u_m, w_1 \ldots w_n\}$. spans $U+W$, is linearly indep.

(e) (1 point) Which of the following definitions, makes $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}^2$ into a surjective linear map?

A. $T(p) = \begin{pmatrix} 0 \\ p(1) \end{pmatrix}$ $T(p_1+p_2) = \begin{pmatrix} 0 \\ p_1+p_2 \end{pmatrix}$ $ax+b$. span$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

B. $T(p) = \begin{pmatrix} p(1) \\ p(1) \end{pmatrix}$ $\begin{pmatrix} p_1 \\ p_1 \end{pmatrix}$ $+ \lambda p$. $\begin{pmatrix} 0 \\ p(1) \end{pmatrix}$ span$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

C. $T(p) = \cancel{\begin{pmatrix} p'(1) \\ p(1) \end{pmatrix}}$ $\begin{matrix} \lambda p(1) \\ \lambda p(1) \end{matrix}$ $(p_1+p_2)'$ — $p_1'\begin{pmatrix} 1 \\ 0 \end{pmatrix} + p_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

D. $T(p) = \cancel{\begin{pmatrix} p-1 \\ p+1 \end{pmatrix}}$ $\begin{matrix} \cancel{2p-1} \\ \cancel{2p+1} \end{matrix}$ span$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$T(\lambda p) = \begin{matrix} \lambda p - 1 \\ \lambda p + 1 \end{matrix}$

2. Give (simple) examples of all of the following situations.

(a) (2 points) Two subspaces U, V of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ such that $U+V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$ but $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \neq U \oplus V$.

$$U = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$ ✓

$$V = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}.$$ ✓

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(b) (2 points) A basis for each of your subspaces U and V above.

$$U: \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$ ✓

$$V: \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$ ✓

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(c) (1 point) A basis for $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ that does not contain either of the bases from the previous part.

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$ ✓

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3. Consider the following maps. Prove or disprove that they are linear and if linear, find the dimension of the kernel (nullspace).

(a) (1 point) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $T(a, b) = a^2 - b^2$.

$T(a,b) = a^2+b^2$.

$T[(a_1+a_2, b_1+b_2)] = (a_1+a_2)^2 + (b_1+b_2)^2$.

$T(a_1,b_1) + T(a_2,b_2) = a_1^2+a_2^2+b_1^2+b_2^2$.

$T[(a_1+a_2, b_1+b_2)] \neq T(a_1,b_1) + T(a_2,b_2)$

not closed under addition. not linear.

(b) (4 points) $R : \mathrm{Mat}_{2\times 2}(\mathbb{C}) \longrightarrow \mathbb{C}^2$ given by

$$R(M) = M \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$R(M_1+M_2) = (M_1+M_2)\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix} = M_1\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix} + M_2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = R(M_1) + R(M_2)$

$R(\lambda M) = (\lambda M)\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \lambda\cdot M\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \lambda \cdot R(M)$

satisfy linearity.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad M\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \underset{2\times 2}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \cdot \underset{2\times 1}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 2a-b \\ 2c-d \end{pmatrix}} = 0.$

$2a = b. \quad 2c = d.$ $\qquad M = \begin{pmatrix} a & 2a \\ c & 2c \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}\right\}.$

$\dim(\ker R) = 2.$

4. (a) (2 points) Let V and W be vector spaces and let $T : V \longrightarrow W$ be a linear map. Prove that the kernel (nullspace), $\ker T \subset V$ is a subspace.

$\ker T = \{T(v) = 0 \mid v \in V\}$. since T is linear map, ✓

$T(v_1+v_2) = T(v_1) + T(v_2)$, if $T(v_1), T(v_2) \in \ker(T)$, $T(v_1+v_2) \in \ker(T)$

$\in W$! not V.

$\forall \lambda \in \mathbb{R}$, $T(\lambda v) = \lambda T(v)$ if $T(v) \in \ker(T)$, $\lambda T(v) \in \ker(T)$ ✗

zero element : $\forall v \in \ker T$. $\exists w \in \ker T$. $w + v = 0 + v = v$.

thus $\ker(T)$ contains the zero element and is closed under addition and scalar multiplication.

① by def of lin. map, $T(0_v) = 0_w$ $0_v \in N(T)$

② $\forall x, y \in N(T)$ $T(x+y) = T(x) + T(y) = 0_w + 0_w = 0_w$.

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3 (b) (3 points) Suppose W_1 and W_2 are two subspaces of a vector space V such that $V = W_1 \oplus W_2$. If B_1 and B_2 are bases of W_1 and W_2 respectively, show that $B_1 \cup B_2$ is a basis for V.

$V = W_1 \oplus W_2 \Rightarrow \begin{cases} W_1 + W_2 = V \\ W_1 \cap W_2 = \{0\} \end{cases}$ ✓

Let $B_1 = \{x_1 \dots x_n\}$ $B_2 = \{y_1 \cdots y_m\}$.

$\forall v \in W_1$. $v = \lambda_1 x_1 + \cdots + \lambda_n x_n$.

$\forall w \in W_2$, $w = \mu_1 y_1 + \cdots + \mu_m y_m$.

$\forall p \in V$. $p = v + w$, $v \in W_1$. $w \in W_2$.

$p = \lambda_1 x_1 + \cdots + \lambda_n x_n + \mu_1 y_1 + \cdots \mu_m y_m$. — spanning. ✓

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since $W_1 \cap W_2 = \{0\}$. $B_1 \cap B_2 = \{0\}$. $B_1 \cup B_2 = \{x_1 \dots x_n, y_1 \dots y_m\}$.

no repetitive elements. ~~Let $\lambda_1 x_1 + \cdots + \lambda_n x_n + \mu_1 y_1 + \cdots + \mu_m y_m = 0$.~~

if $\{x_1 \dots x_n, y_1 \dots y_m\}$ are not linearly independent,

pick $0 = \lambda_1 x_1 + \cdots + \lambda_n x_n + \mu_1 y_1 + \cdots + \mu_m y_m$.

since $\{y_1 \cdots y_m\}$ are linearly independent, $\{x_1 \dots x_n\}$ are linearly independent, $\lambda_1 x_1 + \cdots + \lambda_n x_n = -(\mu_1 y_1 + \cdots + \mu_m y_m)$

LHS $\in W_1$, RHS $\in W_2$. $\Rightarrow$ LHS $\in W_2$ & RHS $\in W_1$.

$W_1 \cap W_2 = \{0\}$. $\Rightarrow$ RHS $=$ LHS $= 0$. $\Rightarrow \lambda_1 = \cdots = \lambda_n = 0$, $\mu_1 = \cdots = \mu_m = 0$

$\Rightarrow \{x_1 \dots x_n, y_1 \dots y_m\}$ are linearly independent.

$\Rightarrow$ it is a basis.