Mathematics 114C, Winter 2019
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There are 12 parts of problems, each worth 10 points for a total of 120 and 100 points count for a "perfect" score; so you can skip a part or two or afford to make a small mistake. Some of these parts are trivial and some are not—so do first those that you can do quickly.

There is a blank page at the end which you can use for scratch, and the page before it has a copy of the transition table for the recursive machine.

Problem 1.	30
Problem 2.	35
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Problem 4.	20
Total:	110

Problem 1. Let

$$C(x,y,z) = \textsf{if } (x=0) \textsf{ then } y \textsf{ else } z = \begin{cases} y, & \text{if } x = 0, \\ z, & \text{otherwise} \end{cases} \qquad (x,y,z \in \mathbb{N})$$

Determine whether each of the following equations is True or False under the indicated assumptions and circle which it is:

(1a) $\boxed{\text{True}}$ or $\boxed{\text{False}}$: for all partial functions $g(\vec{x}), h_1(\vec{x}), h_2(\vec{x})$ on $\mathbb{N}$,

$$C(g(\vec{x}), h_1(\vec{x}), h_2(\vec{x})) = \textsf{if } (g(\vec{x}) = 0) \textsf{ then } h_1(\vec{x}) \textsf{ else } h_2(\vec{x})$$

False, ✓ C is defined by composition, so g, h_1, and h_2 must all $\downarrow$, but the statement on the right only require one of h_1, h_2 to converge*, depending on $g(x)$.

*for convergence

(1b) $\boxed{\text{True}}$ or $\boxed{\text{False}}$: for all partial $g(\vec{x})$ and total $h_1(\vec{x}), h_2(\vec{x})$:

$$C(g(\vec{x}), h_1(\vec{x}), h_2(\vec{x})) = \textsf{if } (g(\vec{x}) = 0) \textsf{ then } h_1(\vec{x}) \textsf{ else } h_2(\vec{x})$$

True, ✓ both sides are equal to $h_1(x)$ if $g(x)\downarrow = 0$, $h_2(x)$ if $g(x)\downarrow \neq 0$, $\uparrow$ if $g(x)\uparrow$.

(1c) $\boxed{\text{True}}$ or $\boxed{\text{False}}$: for all partial $g(\vec{x})$ and $h(\vec{x})$:

$$C(g(\vec{x}), h(\vec{x}), h(\vec{x})) = \textsf{if } (g(\vec{x}) = 0) \textsf{ then } h(\vec{x}) \textsf{ else } h(\vec{x})$$

True. ✓

C =

g \ h	$\uparrow$	$\downarrow$
$\uparrow$	$\uparrow$	$\uparrow$
$\downarrow$	$\uparrow$	$h(x)\downarrow$

= if

Problem 2. Consider the following recursive program (with just one equation) on the partial algebra $\mathbf{N}_0 = (\mathbb{N}, 0, 1, S, Pd)$ of unary natural numbers,

$$(E) \qquad p(x,y) = \text{if } (x=0) \text{ then } 1 \text{ else } p(Pd(x), p(x,y))$$

10 **(2a)** Show the computation of $\overline{p}(0,3)$ by the recursive machine which starts with the state $p\ :\ 0\ 3$, determine if it converges or diverges and determine the value $\overline{p}(0,3)$ if it converges.

$\bar{p}(0,3)$ computed by: $p:0\ 3 \to \text{if } (0=0) \text{ then } 1 \text{ else } p(Pd(0),p(0,3)) : \to$

$1\ p(Pd(0),p(0,3))\ ?\ 0 : \to 1\ p(Pd(0),p(0,3))\ ? : 0 \to 1 : \to : 1$

$\therefore\ \bar{p}(0,3)\downarrow = 1$

10 **(2b)** Show the computation of $\overline{p}(2,3)$ by the recursive machine which starts with the state $p\ :\ 2\ 3$, determine if it converges or diverges and determine the value $\overline{p}(2,3)$ if it converges.

$\bar{p}(2,3)$ computed by: $p:2\ 3 \to \text{if } (2=0) \text{ then } 1 \text{ else } p(Pd(2),p(2,3)) : \to$

$2\ p(Pd(2),p(2,3))\ ?\ 2 : \to 2\ p(Pd(2),p(2,3))\ ? : 2 \to p(Pd(2),p(2,3)) : \to$

$p\ Pd(2)\ p(2,3) : \to p\ Pd(2)\ p\ 2\ 3 : \to p\ Pd(2)\ p\ 2 : 3 \to$

$p\ Pd(2)\ p : 2\ 3 \to \ldots$

Because of the stack discipline lemma, $p:2\ 3 \to p\ Pd(2)\ p : 2\ 3 \Rightarrow$

$p\ Pd(2)\ p : 2\ 3 \to p\ Pd(2)\ p\ Pd(2)\ p : 2\ 3$ $\quad (p:2\ 3 \to \alpha\ p:2\ 3 \Rightarrow p:2\ 3 \to \alpha^n\ p:2\ 3)\ \forall n$.

Therefore $p:2\ 3$ leads to infinitely many longer substrings in its computation and diverges.

(Problem 2 continues on the next page)

(Problem 2 continued from the preceding page)

(2c) Give an explicit definition of the partial function $\overline{p}(x, y)$ computed by the recursive program E.

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$p(x,y) = \text{if } (x=0) \text{ then } 1 \text{ else } p(Pd(x), p(x,y))$

$$\bar{p}(x,y) = \begin{cases} 1, & x = 0 \\ \uparrow, & \text{otherwise} \end{cases}$$

(2d) Prove that the equation $p(x, y) = \text{if } (x = 0) \text{ then } 1 \text{ else } p(Pd(x), p(x, y))$ of the program (E) has exactly one total solution and find it.

5

when $x = 0$, $f(x,y) = 1$

when $x \neq 0$: $f(x,y) = f(x-1, f(x,y))$

if $x-1 = 0$, then $f(x,y) = f(x-1, \ldots) = 1$

$\therefore f(x,y) = 1$ satisfies $p(x,y) = \text{if}(x=0) \text{ then } 1 \text{ else } p(Pd(x), p(x,y))$

for $x = 0$: $f(0,y) = 1$

for $x = n+1$: $f(n+1, y) = \text{if } (n+1) = 0 \text{ then } 1 \text{ else } f(Pd(n+1), f(n+1, y)) =$

$= f(n, f(n+1, y)) = 1$ by induction hyp.

not quite a proof. Because

This is the only total soln, since clearly the only value $f(x,y)$ can return is 1. (By induction, $f(0,y) = 1$, $f(x,y)$ only $\downarrow \Leftarrow f(x-1, f(x,y))$)

(only 1 exit point)

Problem 3. Suppose $g(x), h(u,v), \sigma(x), \tau(x,y)$ are total functions on $\mathbb{N}$.

(3a) Prove that there is exactly one total function $f(x,y)$ such that for all x, y,

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$$f(0,y) = g(y),$$
$$f(x+1,0) = f(x,\sigma(x)),$$
$$f(x+1,y+1) = h(f(x, f(x+1,y)), \tau(x,y))$$

($*$) existence?

let f_1, f_2 be two total functions satisfying ($*$). By induction on x,

$f_1 = f_2$:

$f_1(0,y) = g(y) = f_2(0,y)$, $\forall y$

for $x = n$:

for $y = 0$:

inner induction

$f_1(n+1,0) = f_1(n,\sigma(x)) = f_2(n,\sigma(x)) = f_2(n+1,0)$, by induction

for $y > 0$: $f_1(n+1,y+1) = h(f(x, f(n+1,y)), \tau(x,y)) = \ldots$

(3b) True or false: if g, h, σ, τ are recursive functions, then f is also recursive.

10

True, since an equation can be made for a recursive machine, assuming g, h, σ, τ are recursive.

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(3c) True or false: if g, h, σ, τ are primitive recursive functions, then f is also primitive recursive.

False. (Compare to Ackermann function)

Problem 4. The *iterates* $\phi^i : M \to M$ of a (total) function $\phi : M \to M$ are defined for $i \geq 1$ by the recursion

$$\phi^1(x) = \phi(x), \quad \phi^{i+1}(x) = \phi(\phi^i(x)), \ldots$$

so that $\phi^2(x) = \phi(\phi^1(x)) = \phi(\phi(x)),\ \phi^3(x) = \phi(\phi(\phi(x))), \ldots$

Suppose $\mathbf{M} = (M, 0, 1, g, h)$ is a total algebra with both $g, h : M \to M$ unary, and define the partial function $f : M \rightharpoonup M$ by

$$f(x) = g^n(x) \text{ where } n = (\mu i \geq 1)[h^i(x) = 0]$$

For example, if $h(x) \neq 0, h^2(x) \neq 0, h^3(x) = 0$, then $f(x) = g^3(x)$, and if $h^i(x) \neq 0$ for all i, then $f(x)\uparrow$.

(4a) Specify a recursive program of $\mathbf{M}$ which computes $f(x)$.

Careful: M is an arbitrary set, not necessarily $\mathbb{N}$, and so we cannot use functions on $\mathbb{N}$.

$f(x) = p(x, h(x))$

$p(x, y) = \text{if } (y = 0) \text{ then } g(x) \text{ else } p(g(x), h(y))$

10

(Problem 4 continues on the next page)

(Problem 4 continued from the preceding page)

(4b) Prove the claim you made in part (4a), that the recursive program you specified actually computes the partial function $f(x)$.

Proof 10 by induction over n: $\left(f(x) = g^n(x)\right)$

$h^n(x) = 0$

for $n = 1$:

$f(x) = p(x, h(x)) = \text{if } (h(x) = 0) \text{ then } g(x) \text{ else } p(g(x), h(h(x))) =$

$= g(x)$ ✓

for $n+1$, assuming $f(x) = p(x, h(x)) = g^n(x)$, when $n = \mu n(h^n(x) = 0)$;

$f(x) = p(x, h(x)) = \text{if } (h^n(x) = 0) \text{ then } g^n(x) \text{ else } p(g^n(x), h^{n+1}(x)) =$

$= \text{if } (h^{n+1}(x) = 0) \text{ then } g(g^n(x)) \text{ else } p(g(g^n(x)), h(h^{n+1}(x))) =$

$= g(g^n(x)) = g^{n+1}(x)$.

$\therefore \forall n,\ f(x) = g^n(x) \iff n = \mu n(h^n(x) = 0)$

Since n is unbounded, if $\nexists n$, $f(x)\uparrow$ ☒