Math 110BH
Winter 2020

Final Exam

Instructions: You have 3 hours to complete this exam. Per departmental policies, you are given a 24 hours time window to work on the exam and submit it: from Friday, March 20, 1am PST, to Saturday, March 21, 1am PST.

You can use the official course textbook, Elman's notes, your own notes from lecture and discussion section, as well as your graded HW sets and midterms. No other source (paper source, electronic source, or any person other than you) is allowed.

You can either print this booklet and solve the exam questions on this printout, or you can use separate paper sheets, in which case please write clearly on top of each page which Question and Part you are solving. In either case, you need to scan or take photos of your solutions and upload it back to CCLE as a single pdf file before the 1am Saturday deadline. Please order your solutions in the same order as the questions on this booklet.

For full credit show all of your work legibly, preferably using a pen. Illigible solutions will receive no credit. Unless instructed otherwise, you need to justify your answers.

Full Name: ______________________________
Student ID number: ________________________

Signature: ______________________________

Question	Points	Score
1	14	
2	10	
3	15	
4	15	
5	16	
6	12	
7	15	
8	18	
9	23	
10	12	
Total:	150	

Problem 1.

Let R be a commutative ring with 1. Recall that an R-module M is called *simple* if it admits no nonzero proper R-submodules.

(a) [5pts.] Suppose that M_1, M_2 are simple R-modules. Suppose that $\phi : M_1 \longrightarrow M_2$ is a nonzero R-module homomorphism. Prove that M_1 and M_2 are isomorphic.

(b) [9pts.] Give an example of an R-module N which is not simple, but such that N is not the internal direct sum of two nonzero, proper, R-submodules (in other words, if there exists R-submodules N_1, N_2 such that $N = N_1 \oplus N_2$, then $N_1 = 0$ and $N_2 = N$ or vice versa).

Problem 2. *10pts.*

Let R be a commutative ring with 1 and let M be a Noetherian R-module.

Show that any injective R-module homomorphism $f : M \longrightarrow M$ is an isomorphism.

Problem 3.

[This problem continues onto the next page.]

Let R be a ring with 1. Define the *center* of R to be $Z(R) = \{r \in R \,|\, rs = sr \,\forall s \in R\}$.

(a) [3pts.] Show that $Z(R)$ is a subring of R.

(b) [6pts.] Let now G be a group, and recall that $Z(G) = \{g \in G \,|\, gh = hg \,\forall h \in G\}$ is the center of the group G. Show that the injection $\phi : Z(G) \hookrightarrow G$ extends $\mathbf{Q}$-linearly to a ring monomorphism $\Phi : \mathbf{Q}[Z(G)] \hookrightarrow \mathbf{Q}[G]$.

[This problem continues from the previous page.]

Recall that you proved that $\Phi : \mathbf{Q}[Z(G)] \hookrightarrow \mathbf{Q}[G]$ is a ring monomorphism.

(c) [6pts.] Is the image of Φ equal to $Z(\mathbf{Q}[G])$? Prove that it is so, or give a counterexample.

Problem 4. *15pts.*

Classify all abelian groups of size 120 up to isomorphism, using your favorite structure theorem for finitely generated modules over a PID. Make sure to explain your reasoning.

Problem 5.

(a) [8pts.] Describe all the possible rational canonical forms of 2×2 matrices over $\mathbf{F}_2$, the field with 2 elements.

(b) [8pts.] Describe all the possible rational canonical forms of 2×2 matrices over $\mathbf{Q}$.

Problem 6.

Let R be a ring with 1. An element $e \in R$ is called an *idempotent* if $e^2 = e$.

(a) [4pts.] Suppose that $e \in R$ is an idempotent. Show that $1 - e$ is an idempotent too.

(b) [8pts.] Let M be a left R-module, and suppose that $\phi \in \operatorname{Hom}_R(M, M)$ is an idempotent.[1] Show that $M = \phi(M) \oplus (1_S - \phi)(M)$.

[1] The multiplication in the ring $S = \operatorname{Hom}_R(M, M)$ is given by composition of R-module homomorphisms.

Problem 7. *15pts.*

Let R be a PID, and let $a, b \in R$.

Let d be any greatest common divisor of a and b. Show that $R/(a) \otimes_R R/(b) \cong R/(d)$.

Problem 8.

Let $p(x) = x^6 - 1$.

(a) [6pts.] Factor $p(x)$ over $\mathbf{Z}$.

(b) [6pts.] Factor $p(x)$ over $\mathbf{Z}/2\mathbf{Z}$.

(c) [6pts.] Factor $p(x)$ over $\mathbf{Z}/3\mathbf{Z}$.

Problem 9.

[This problem continues onto the next page.]

Let R be a PID and let p, q be two prime elements of R, not associate to one another.

(a) [5pts.] Let F be a fraction field of R. Show that F/R is a torsion R-module under the action by left multiplication.

(b) [6pts.] Describe $(F/R)_p$, the p-primary part of F/R.

[This problem continues from the previous page.]

Recall that R is a PID and p, q are two non-associate prime elements of R.

(c) [5pts.] Let M be a p-primary R-module and N is a q-primary R-module. Show that $\mathrm{Hom}_R(M, N) = 0$.

(d) [7pts.] Let M be a finitely generated R-module. Show that $\mathrm{Hom}_R(M, F/R)$ is isomorphic to $\mathrm{Tor}(M)$ as R-modules.

Problem 10. *12pts.*

Let R be a commutative ring with 1, and suppose that M and N are free R-modules.

Is $\mathrm{Hom}_R(M, N)$ a free R-module? Prove that it must be, or find a counterexample.

Use this page for any additional work. Make sure to clearly state which problem you are solving on this page.

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