Math 110AH Midterm

There are five problems. Do the first four problems today.
The fifth problem is due Monday 13 November 17.

1. (15 points) Give examples of each of the following (No justification required):
   a. A non-abelian group of order 20.
   b. A non-abelian infinite group.
   c. A group with at least 32 elements having no proper normal subgroups. (Recall a subgroup is called *proper* if it is not the whole group or the group consisting solely of the identity (unity) element.)
   d. A homomorphism of groups that is neither injective or surjective.
   e. A group G that has a proper non-abelian normal subgroup and a subgroup H of G that is non-abelian and normal.

2. (25 points) Do all of the following:
   a. Accurately state the Fundamental Theorem of Arithmetic and prove it.
   b. Accurately state Lagrange's Theorem and prove it.
   c. Accurately state a total of three corollaries of Lagrange's Theorem.

3. (40 points) For each of the following fourteen groups clearly label the group and write
   - its order
   - whether it is cyclic, abelian but not cyclic, or non abelian.

   Then
   - list which ones are isomorphic

   (No justification is required):

   a. $\mathbf{Z}/6\mathbf{Z}$ (under addition).
   b. $\mathbf{Z}/8\mathbf{Z}$ (under addition).
   c. $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ (under component-wise addition).
   d. $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ (under component-wise addition).
   e. $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ (under component-wise addition).
   f. $(\mathbf{Z}/8\mathbf{Z})^\times$ (under multiplication).
   g. $(\mathbf{Z}/9\mathbf{Z})^\times$ (under multiplication).
   h. $(\mathbf{Z}/4\mathbf{Z})^\times \times (\mathbf{Z}/6\mathbf{Z})^\times$ (under component-wise multiplication).
   i. S_3
   j. S_4
   k. D_3
   l. D_4
   m. $GL_2(\mathbf{Z}/2\mathbf{Z})$
   n. $SL_2(\mathbf{Z}/3\mathbf{Z})$

4. (30 points) Write whether each of the following ten statements is true or false (or leave unanswered). Your score is 3pts$\times$ #(correct answers) – 3pts$\times$ #(incorrect answers). **Guessing can be very hazardous.**

   a. Let $a, b \in \mathbf{Z}$. Then $\gcd(a, b) = 1$ if and only if whenever $c \in \mathbf{Z}$ and $a \mid bc$, then $a \mid c$.

   b. Let $m > 1$ be an integer not divisible by 15. Then the congruence $10x \equiv 5 \bmod m$ always has a solution in integers.

   c. Let $n > 1$ be an integer. Then for any integer x the congruence $x^n \equiv x \bmod n$ is satisfied.

   d. Let a, b, c be positive integers. Then the (diophantine) equation $ax + by = c$ has only finitely many solutions (if any) in positive integers x, y.

   e. Let G be a cyclic group of order $m > 1$. Then $G = \langle a \rangle$ for $\phi(m)$ distinct elements a in G, where ϕ is the Euler ϕ-function.

   f. Let G be a group. The *center* of G is defined to be $\{x \in G \mid xy = yx \text{ for all } y \in G\}$. Then the center of G is a normal subgroup of G. Moreover, the center of G is all of G if and only if G is abelian.

   g. The group of real numbers $\mathbf{R}$ under addition contains the integers $\mathbf{Z}$ as a normal subgroup and there is a bijection from the set of cosets $\mathbf{R}/\mathbf{Z}$ to the circle group $\mathbf{T} := \{z \in \mathbf{C} \mid |z| = 1\}$.

   h. The automorphism group of an abelian group G, i.e., the group

$$\mathrm{Aut}(G) := \{\sigma : G \to G \mid \sigma \text{ an isomorphism}\}$$

   under composition is an abelian group.

   i. There exist at least three non-isomorphic groups G having $|\mathrm{Aut}(G)| = 2$.

   j. Let G be a group of order pq with p and q primes (but not necessarily distinct). Then G contains a subgroup of order p or a subgroup of order q.

5. (100 points) Do part 4 at home.

   In addition,

   - You must give a proof or counterexample to each statement.
   - You must do it yourself.
   - If you use a book or the web, results used must be proved if not proven in class or prior homework. If you are copying a proof from a book or a file on the web (or essentially copying a proof) write the book and page number or web address and justify the details (some which may not be written down in the book or on the website). Cutting and pasting is not allowed.

Your solutions for Part 5 is due Monday 13 Nov 17.
Make an extra copy of your solutions.