Exam 3

(Q-1) Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangle.

Solution. If x and y are the sides of the two squares with combined area 1, then $x^2+y^2=1$. Suppose without loss of generality that $x \geq y$. Then the shorter side of a rectangle containing both squares without overlap must be at least x, and the longer side must be at least $x+y$. Hence the desired value of A is the maximum of $x(x+y)$ subject to the constraints $x^2+y^2=1$ and $x \geq y > 0$. To find this maximum, substitute $x=\cos\theta, y=\sin\theta$ with $\theta \in (0, \pi/4]$. Then

$$\cos^2\theta + \sin\theta\cos\theta = \frac{1}{2}(1+\cos 2\theta + \sin 2\theta) = \frac{1+\sqrt{2}\cos(2\theta - \pi/4)}{2} \leq \frac{1+\sqrt{2}}{2},$$

with equality for $\theta = \pi/8$, so the least A is $(1+\sqrt{2})/2$.

(Q-2) For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x)=\pi(y)$ and $\pi'(x)=\pi'(y)$. (A partition of a subset S is a collection of disjoint subsets (parts) whose union is S.)

Solution. For a given π, no more that three different values of $\pi(x)$ are possible. (Four would require one part each of size at least $1,2,3,4$, which is already twoo many elements.) If no such x, y exist, each pair $(\pi(x), \pi'(x))$ occurs for at most one x, and since there are at most 3×3 possible pairs, each must occur exactly once. Moreover, there must be three different values of $\pi(x)$, each occurring 3 times. However, any given value of $\pi(x)$ occurs $k\pi(x)$, where k is the number of distinct parts of that size. Thus $\pi(x)$ can occur 3 times only if $\pi(x)$ equals 1 or 3, but we have three distinct values occuring 3 times, contradiction.

(Q-3) Let G be a group with identity e and $\phi\colon G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism.

Solution. To be a homomorphism $\psi(e) = e$ and $\psi(xy) = \psi(x)\psi(y)$ for all x, y. We can ensure $\psi(e) = e$ by setting $a = \phi(e)^{-1}$. From hypothesis, we get

$$\phi(g)\phi(e)\phi(g^{-1}) = \phi(e)\phi(g)\phi(g^{-1}),$$

so $\phi(g)$ commutes with $\phi(e)$ for all g. Also from the hypothesis

$$\phi(x)\phi(y)\phi(y^{-1}x^{-1}) = \phi(e)\phi(xy)\phi(y^{-1}x^{-1}).$$

Since $\phi(e)$ commutes with anything in the image of ϕ, $\phi^{-1}(e)$ does too, so we deduce

$$\psi(x)\psi(y) = \phi(e)^{-1}\phi(x)\phi(e)^{-1}\phi(y) = \phi^{-1}(e)\phi(xy) = \psi(xy).$$

(Q-4) The function $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0)=2, f'(0)=-2$ and $f(1)=1$. Prove that there exists a real number $c \in (0,1)$ for which

$$f(c)\cdot f'(c) + f''(c) = 0.$$

Solution. Define the function

$$g(x) = (f(x))^2/2 + f'(x).$$

We want a root of g'. Since $g(0)=0$, it is enough to prove that exists a real number $0 < \eta \leq 1$ for which $g(\eta)=0$. If f is never zero, let

$$h(x) = x/2 - 1/f(x).$$

Because $h(0) = h(1) = -1/2$, there exists a real number $0 < \eta < 1$ for which $h'(\eta) = 0$. But $g = f^2 \cdot h'$, and we are done. If f has at least one zero, let z_1 be the first one and z_2 be the last one. (The set of the zeros is closed.) So $0 < z_1 \leq z_2 < 1$. The function is positive on the intervals $[0, z_1)$ and $(z_2, 1]$; this implies that $f'(z_1) \leq 0$ and $f'(z_2) \geq 0$. Then $g(z_1) = f'(z_1) \leq 0$ and $g(z_2) = f'(z_2) \geq 0$, and there exists a real number $\eta \in [z_1, z_2]$ for which $g(\eta) = 0$.

-5) On average, how many times do you have to roll a die before all six different numbers have turned

Solution. If one repeats an experiment whose probability of success is p, then the mean waiting ti for success is

$$\sum_{n=1}^{\infty} n(1-p)^{n-1}p = 1/p.$$

Now we break up the given problem into six stages, and by addition of expectations, the mean ti to complete all the stages is the sum of the individual mean times of the stages. Define "stage k" be the period during which we have seen $k-1$ different numbers and are waiting to see the k-th. T probability of success during stage k is just the number of numbers we haven't seen, namely $n-k+$ divided by 6; so the mean waiting time of stage k is $6/(n-k+1)$. It follows the mean time for t whole process is

$$\frac{6}{6}+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1} = 14.7$$

-6) Let $f\colon (0,1) \to [0,\infty)$ be a function that is zero except at the (countably many) distinct poi $a_1, a_2, \ldots$. Let $b_n = f(a_n)$. Prove that if

$$\sum_{n=1}^{\infty} b_n < \infty,$$

then f is differentiable at at least one point $x \in (0,1)$.

Solution. We first construct a sequence c_n of positive numbers such that $c_n \to \infty$ and $\sum_{n=1}^{\infty} c_n b_n$ 1/2. Let $B = \sum_{n=1}^{\infty} b_n$, and for each $k = 0, 1, \ldots$ denote by N_k the first positive integer for which

$$\sum_{n=N_k}^{\infty} b_n \le \frac{B}{4^k}.$$

Now set $c_n = \frac{2^k}{5B}$ for each n, $N_k \le n < N_{k+1}$. Then we have $c_n \to \infty$ and

$$\sum_{n=1}^{\infty} c_n b_n = \sum_{k=0}^{\infty}\sum_{n=N_k}^{N_{k+1}} c_n b_n \le \sum_{k=0}^{\infty} \frac{2^k}{5B} \sum_{n=N_k}^{\infty} b_n \le \sum_{k=0}^{\infty} \frac{2^k}{5B}\cdot\frac{B}{4^k} = \frac{2}{5}.$$

Consider the intervals $I_n = (a_n - c_n b_n, a_n + c_n b_n)$. The sum of their lengths is $2\sum c_n b_n < 1$, thus the exists a point $x_0 \in (0,1)$ which is not contained in any I_n. We show that f is differentiable at x_0 ar $f'(x_0) = 0$. Since x_0 is outside of the intervals I_n, $x_0 \neq a_n$ for any n, so $f(x_0) = 0$. For arbitra $x \in (0,1) \setminus \{x_0\}$, if $x = a_n$ for some n, then

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| = \frac{f(a_n)}{|a_n - x_0|} \le \frac{b_n}{c_n b_n} = \frac{1}{c_n},$$

otherwise $\frac{f(x)-f(x_0)}{x-x_0} = 0$. Since $c_n \to \infty$, this implies for arbitrary $\epsilon > 0$, there are only finitely mar $x \in (0,1) \setminus \{x_0\}$ for which

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| < \epsilon$$

does not hold, and we are done.