Midterm

Version A

Last Name: ____________________

First Name: ____________________

Student ID: ____________________

Signature: ____________________

Lab Section:

1A (Mon 9–11)	**1B** (Mon 11–1)	**1C** (Mon 2–4)
1D (Mon 4–6)	**1E** (Tue 9–11)	**1F** (Tue 11–1)
1G (Tue 1–3)	**1J** (Wed 11–1)	**1K** (Wed 2–4)
1M (Thu 10–12)	**1N** (Thu 12–2)	**1O** (Thu 2–4)

Instructions: Do not open this exam until instructed to do so. You will have 90 minutes to complete the exam. Please print your name and student ID number above, and circle your lab section. You may not use books, notes, or any other material to help you. You may use a calculator, but not a programmable or graphing calculator. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work, including the backs of the pages. If you need more scratch paper, please ask one of the proctors.

Please do not write below this line.

Question	Points	Score
1	12	
2	8	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. A farmer cultivates bees for honey, and has many acres of wildflowers from which the bees can collect honey. These both grow so successfully that he eventually decides to get some goats to graze on the wildflowers. Consider a model for these three populations based the following assumptions:

   - Since the bees rely on the wildflowers for food, the per-capita birth rate of the bees is proportional to the amount of wildflowers, with a proportionality constant of 0.02.
   - Since the wildflowers rely on the bees for pollination, the per-capita reproduction rate of the wildflowers is proportional to the bee population, with a proportionality constant of 0.1.
   - The bees have a natural per-capita death rate of 40% per month.
   - The wildflowers have a natural per-capita death rate of 22% per month.
   - The goats eat the wildflowers at a rate proportional to the number of goats times the number of wildflowers, with a proportionality constant of 0.04.
   - Since the goats rely on the wildflowers for food, the per-capita birth rate of the goats is proportional to the number of wildflowers, with a proportionality constant of 0.01.
   - Finally, the goats have a per-capita death rate of 1% per month.

   (a) (8 points) Write down a system of differential equations (with variables B, W, and G) for this model.

Question 1 continues on the next page. . .

(b) (4 points) Describe both a positive and a negative feedback loop in this model. *(Hint: Diagram(s) might help!)*

2. (8 points) In the wilderness of northern California, elk (E) and deer (D) compete for resources. Set up a model of these two populations based on the following assumptions. Where you need a coefficient or proportionality constant and one is not given, just insert a parameter into your model (such as k, r, c, or whatever letters you wish).

   - In the absence of crowding/competition, deer have a natural per-capita birth rate of 25%, and a natural per-capita death rate of 12%.
   - Likewise, in the absence of crowding/competition, elk have a natural per-capita birth rate of 12%, and a natural per-capita death rate of 6%.
   - Crowding adversely affects the elk population: when two elk attempt to forage in the same small area, there is not enough food for both, and this decreases the growth rate of their population.
   - Crowding within the deer population does not affect the deer.
   - Competition occurs between the elk and the deer: when an elk and a deer find themselves in the same area, there is not enough food for both, and this decreases the growth rate of *both* populations. Assume this hurts the deer population twice as much as it hurts the elk.

3. (10 points) The levels of glucose (G) and insulin (I) in the human body can be modeled by the differential equations

$$\begin{cases} G' = 10 - 0.02GI - 0.1G \\ I' = \frac{G}{5+G} - 0.01I \end{cases}$$

Suppose that initially the level of glucose is $5\,\frac{\text{mmol}}{\text{L}}$ and the level of insulin is $150\,\frac{\text{pmol}}{\text{L}}$. Use Euler's method with a step size of $\Delta t = 0.1\,\text{s}$ to approximate the state $0.2\,\text{s}$ later.

4. (a) (5 points) Assume you have a differential equation model with three variables[1]. State what the Fundamental Theorem on Existence and Uniqueness of Solutions to Ordinary Differential Equations says about this model.

   (b) (5 points) Suppose that, in a differential equation model, a trajectory begins in some state, and it later reaches the exact same state. What can you say about the behavior of this trajectory? Justify your answer (briefly).

[1] And assume, as we always do, that this differential equation satisfies the hypothesis of the Fundamental Theorem, i.e., that as a function from state space to tangent space it is "continuously differentiable".

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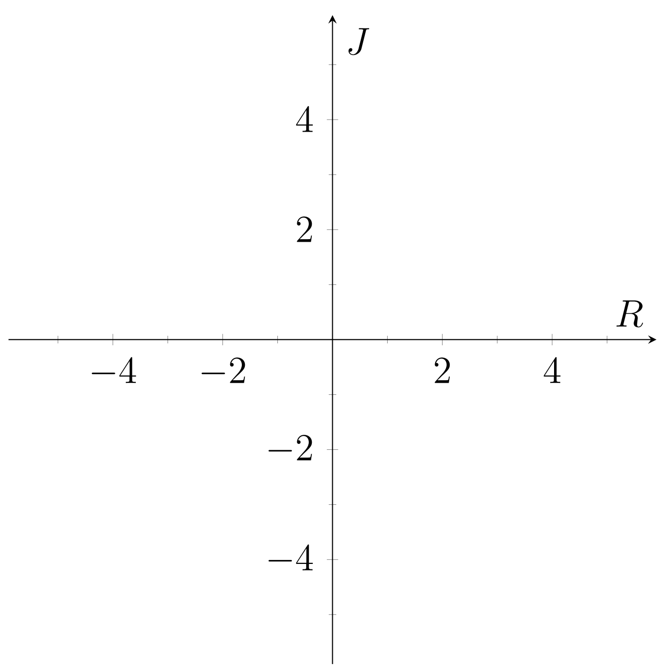
5. (10 points) Let R be Romeo's love for Juliet (or hate if negative), and J be Juliet's love (or hate) for Romeo. Assume these are governed by the differential equations

$$\begin{cases} R' = 0.2R + 0.4J \\ J' = 0.5J - 0.1R^2 \end{cases}$$

Sketch the vector field of this system on the axes provided below. (Five vectors is enough, given the time constraints. Try to spread them out. And try to draw them as accurately as possible!)



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6. Let f be a function of x.

   (a) (3 points) Explain, in both words and mathematical expressions, what the average rate of change of f with respect to x is.

   (b) (3 points) Explain what the instantaneous rate of change of f at $x = x_0$ means, and how you can find it, in general.

   (c) (4 points) Now suppose you know that $f'(x) = \sqrt{x^3 - 2}$. If $f(3) = 8$, then approximately what is $f(2.8)$?