ECON 101
Professor Vogel
Midterm #2 Solutions

Question 1 Solution (50 points). An SPNE is a set of strategies that represent a Nash equilibrium starting from any proper subgame. There are three proper subgames in the game. One proper subgame corresponds to the game that follows if Player 1 has choosen L (in the game tree, this proper subgame starts from the top of the two decision nodes for Player 2). A second proper subgame corresponds to the game that follows if Player 1 has choosen R (in the game tree, this proper subgame starts from the bottom of the two decision nodes for Player 2). And, of course, the full game is itself a proper subgame (in the game tree, this corresponds to the game that starts from Player 1's decision node). To find all SPNE, we must use backwards induction. (*Students needn't write out this intro. They just need to focus on the relevant cases.*)

First, let's analyze the proper subgame corresponds to the game that follows if Player 1 has choosen L. In this case, there is a single pure strategy Nash equilibrium to the game, and in this Nash equilibrium Player 2 chooses A and Player 3 chooses C. Thus, in any SPNE in which Player 1 has choosen L, Player 2 must choose A and Player 3 must choose C.

	C	D
A	4,**1**,**1**	11,**3**,0
B	-6,0,**3**	8,2,2

Figure 1: Payoff matrix if Player 1 chooses L

Second, let's analyze the proper subgame corresponds to the game that follows if Player 1 has choosen R. In this case, there are two pure strategy Nash equilibrium to the game. In one pure strategy Nash equilibrium Player 2 chooses A and Player 3 chooses C. In the other pure strategy Nash equilibrium Player 2 chooses B and Player 3 chooses D.

	C	D
A	0,**2**,**1**	22,0,0
B	0,0,0	8,**1**,**2**

Figure 2: Payoff matrix if Player 1 chooses R

In summary, there are two possible sets of strategies for Players 2 and 3 in an SPNE. These can be written as:

1. Player 2: $\{A|L, A|R\}$; Player 3: $\{C|L, C|R\}$
2. Player 2: $\{A|L, B|R\}$; Player 3: $\{C|L, D|R\}$

We now turn to the full game. If the strategies of Players 2 and 3 are $\{A|L, A|R\}$ and $\{C|L, C|R\}$, then Player 1's utility if choosing L is 4 and Player 1's utility if choosing R is 0. Hence, in this case, Player 1's best response is to choose L. If the strategies of Players 2 and 3 are $\{A|L, B|R\}$ and $\{C|L, D|R\}$, then Player 1's utility if choosing L is 4 and Player 1's utility if choosing R is 8. Hence, in this case, Player 1's best response is to choose R. Thus, the two SPNE can be written as

1. Player 1: L; Player 2: $\{A|L, A|R\}$; Player 3: $\{C|L, C|R\}$
2. Player 1: R; Player 2: $\{A|L, B|R\}$; Player 3: $\{C|L, D|R\}$

Question 2 Solution (40 points). An SPNE is a set of strategies that represent a Nash equilibrium starting from any proper subgame. All proper subgames can be grouped into two. In first, some player has not cooperated in the past. In the second, all players have cooperated in every previous stage. (*Students needn't write out this intro. They just need to focus on the two cases.*)

(20 points) First, let's analyze the proper subgame starting in some stage s in which some player has not cooperated in the past. In this case, Player 1's strategy calls for her to play B in every stage $t \geq s$ and Player 2's strategy calls for him to play L in every stage $t \geq s$. Since neither player's action in stage s affects future actions, these strategies are a Nash equilibrium to the game starting in stage s if and only if (B,L) is a Nash equilibrium to the stage game. The most general, or weakest conditions under which (B,L) is a Nash equilibrium to the stage game are $x_3 \geq x_1$ and $y_3 \geq y_4$.

(20 points) Second, let's analyze the proper subgame starting in some stage s in all players have cooperated in every previous stage. In this case, holding Player 2's strategy fixed, if Player 1 does not deviate her utility starting in stage s is

$$V_1^C = x_2 + x_2\delta + x_2\delta^2 + ... = x_2\left(1 + \delta + \delta^2 + ...\right) = x_2 \sum_{t=0}^{\infty} \delta^t = x_2 \frac{1}{1-\delta}$$

If Player 1 deviates, her utility starting from stage s is

$$V_1^D = x_4 + x_3\delta + x_3\delta^2 + ... = x_4 + x_3\left(\delta + \delta^2 + ...\right) = x_4 + x_3 \sum_{t=1}^{\infty} \delta^t = x_4 + x_3 \frac{\delta}{1-\delta}$$

Similarly, holding Player 1's strategy fixed, if Player 2 does not deviate his utility starting in stage s is

$$V_2^C = y_2 + y_2\delta + y_2\delta^2 + ... = y_2\left(1 + \delta + \delta^2 + ...\right) = y_2 \sum_{t=0}^{\infty} \delta^t = y_2 \frac{1}{1-\delta}$$

If Player 2 deviates, his utility starting from stage s is

$$V_2^D = y_1 + y_3\delta + y_3\delta^2 + ... = y_1 + y_3\left(\delta + \delta^2 + ...\right) = y_1 + y_3 \sum_{t=1}^{\infty} \delta^t = y_1 + y_3 \frac{\delta}{1-\delta}$$

Player 1 has no incentive to deviate unilateraly if

$$V_1^C \geq V_1^D \iff x_2 \frac{1}{1-\delta} \geq x_4 + x_3 \frac{\delta}{1-\delta} \iff x_2 \geq x_4(1-\delta) + x_3\delta$$
$$\iff (x_4 - x_3)\,\delta \geq x_4 - x_2 \iff \delta \geq \frac{x_4 - x_2}{x_4 - x_3}$$

where I used the fact that $x_4 > x_3$ when dividing by $x_4 - x_3$ (*students needn't mention this*). Player 2 has no incentive to deviate unilateraly if

$$V_2^C \geq V_2^D \iff y_2 \frac{1}{1-\delta} \geq y_1 + y_3 \frac{\delta}{1-\delta} \iff y_2 \geq y_1(1-\delta) + y_3\delta$$
$$\iff (y_1 - y_3)\,\delta \geq y_1 - y_2 \iff \delta \geq \frac{y_1 - y_2}{y_1 - y_3}$$

where I used the fact that $y_1 > y_3$ when dividing by $y_1 - y_3$ (*students needn't mention this*). Hence, we also require that $\delta \geq \frac{y_1 - y_2}{y_1 - y_3}$ and $\delta \geq \frac{x_4 - x_2}{x_4 - x_3}$.

Summary. In summary, we require $\delta \geq \frac{y_1 - y_2}{y_1 - y_3}$, $\delta \geq \frac{x_4 - x_2}{x_4 - x_3}$, $x_3 \geq x_1$, and $y_3 \geq y_4$. (*students needn't summarize if they got the relevant conditions*)

Question 3a Solution (5 points). One pure strategy x strictly dominates another pure strategy y for Player i if Player i receives strictly higher utility when playing x than when playing y, whatever the strategies of the other players (*students needn't define this*). T strictly dominates B for Player 1 if $x_1 > x_3$ and $x_2 > x_4$. L strictly dominates R for Player 2 if $y_1 > y_2$ and $y_3 > y_4$.

Question 3b Solution (5 points). Player 1 is willing to strictly mix between T and B given Player 2's strategy if $\mathbb{E}\left[U_1^T(r)\right] = \mathbb{E}\left[U_1^B(r)\right]$ and Player 2 is willing to strictly mix between L and R given Player 1's strategy if $\mathbb{E}\left[U_2^L(t)\right] = \mathbb{E}\left[U_2^R(t)\right]$, where $\mathbb{E}\left[U_i^X(y)\right]$ is the expected utility for Player i if playing pure strategy X given the other player's strategy of mixing with probability y (*students needn't state all this*).

We have $\mathbb{E}\left[U_1^T(r)\right] = x_1(1-r) + x_2 r$, $\mathbb{E}\left[U_1^B(r)\right] = x_3(1-r) + x_4 r$, $\mathbb{E}\left[U_2^L(t)\right] = y_1 t + y_3(1-t)$, and $\mathbb{E}\left[U_2^R(t)\right] = y_2 t + y_4(1-t)$. Hence, we have

$$\mathbb{E}\left[U_1^T(r)\right] = \mathbb{E}\left[U_1^B(r)\right] \iff x_1(1-r) + x_2 r = x_3(1-r) + x_4 r$$
$$\iff r\,(x_1 - x_2 - x_3 + x_4) = x_1 - x_3 \iff r^* = \frac{x_1 - x_3}{x_1 - x_2 - x_3 + x_4}$$

and we have

$$\mathbb{E}\left[U_2^L(t)\right] = \mathbb{E}\left[U_2^R(t)\right] \iff y_1 t + y_3(1-t) = y_2 t + y_4(1-t)$$
$$\iff t\,(y_1 - y_2 - y_3 + y_4) = y_4 - y_3 \iff t^* = \frac{y_4 - y_3}{y_1 - y_2 - y_3 + y_4}$$

(Students needn't write that we must have $r^* \in (0,1)$ and $t^* \in (0,1)$ or provide conditions on the xs and ys that make these conditions hold.)