SOLUTIONS

UCLA Department of Electrical Engineering
EE2 – Physics for Electrical Engineers
Spring 2013
Midterm, May 2 2013, (1:45 minutes)

Name _____ Student number_____

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at least you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

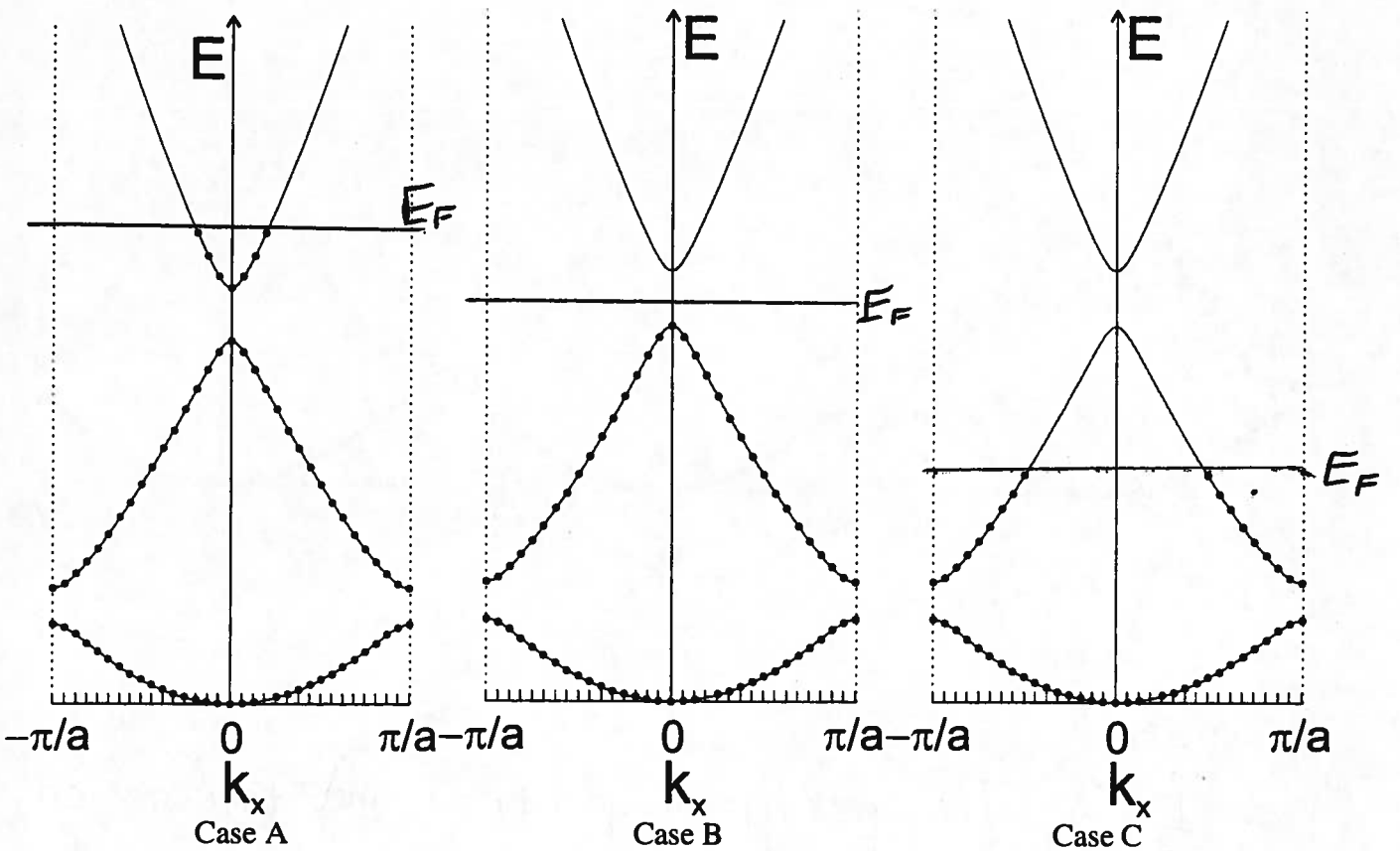
	Topic	Max Points	Your points
Problem 1	Band diagram and electrical conduction	40	
Problem 2	Electron wavefunctions	30	
Problem 3	Fermi-Dirac distribution	30	
Total		100	

This page intentionally left blank.

1. (40 points) Band diagram and electrical conduction

Consider the following band structure diagram for electrons in a 1D crystal with lattice constant *a*. Solid circles indicate occupied electron states. Empty circles in the valence band indicates an unoccupied electron state (i.e. a hole).

(a) (20 point) Consider the following band structure diagrams for a 1D crystal at temperature T=0 K. Sketch the Fermi level E_F on each diagram. For each diagram, is the material a conductor or an insulator?



At T=0

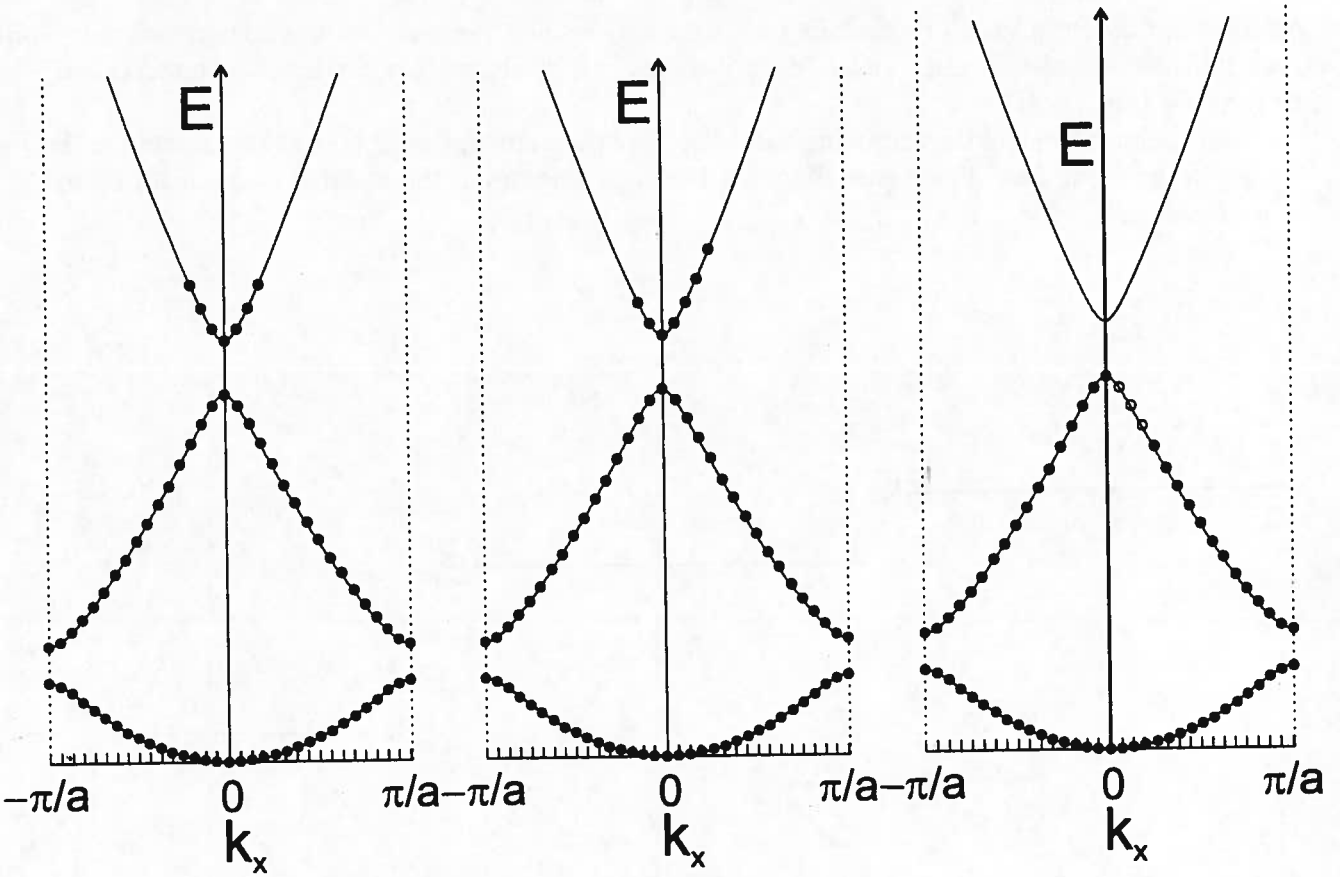
Case A Conductor

Case B Insulator

Case C Conductor.

If E_F is inside the band, the material is a conductor.

(b) (20 points) For each case shown below in the bandstructure diagrams, state whether the current I flowing in the 1D crystal is zero, positive (+x direction), or negative (-x direction).



Case A Case B Case C

Case A $I=0$ The system is in equilibrium, and there are an equal number of electrons populating $+k$ and $-k$ states.

Case B $I<0$ Non-equilibrium distribution with conduction band electrons occupying extra $+k$ states. Average velocity $\langle v_x \rangle > 0$. Electrons have negative charge so $I<0$.

Case C $I<0$ Conduction band is empty but there are more valence band electrons occupying states with $k<0$ than $k>0$. However in the valence band states with $k<0$ have group velocity >0, so $\langle v_x \rangle > 0$.

2. Electron wavefunctions (30 points)

Consider a free electron system in a 1D crystal of length L with energy dispersion $E=\hbar^2k^2/2m_n^*$.

(a) (10 points) If the electron has a wavefunction $\psi(x,t=0)=\frac{1}{\sqrt{L}}e^{-jk_0x}$, what is the expectation value (i.e. average measured value) for the momentum p?

$\langle p\rangle = \hbar\langle k\rangle = -\hbar k_0$: Plane wave has well defined momentum

$$\langle p\rangle = -j\hbar\int_0^L dx\ \psi^*(x)\frac{\partial}{\partial x}\psi(x) = -j\hbar\int_0^L dx\ \frac{1}{\sqrt{L}}e^{jk_0x}\frac{\partial}{\partial x}\frac{1}{\sqrt{L}}e^{-jk_0x}$$

$$= \frac{-j\hbar}{L}(-jk_0)\int_0^L dx = -\hbar k_0$$

The formula given on the sheet had a typo for $\langle p\rangle$, so full credit will be given for Part (a).

(b) (10 points) Consider two electrons:

electron A has a wavefunction $\psi_A(x,t=0)=\frac{1}{\sqrt{a\sqrt{2\pi}}}e^{-x^2/4a^2}e^{jk_0x}$,

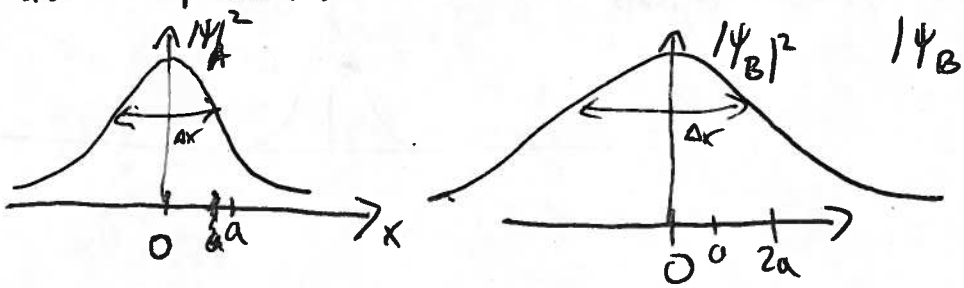
and electron B has a wavefunction $\psi_B(x,t=0)=\frac{1}{\sqrt{a\sqrt{2\pi}}}e^{-x^2/16a^2}e^{j4k_0x}$, where $a=10^{-9}$ m.

Which electron has a larger uncertainty in position? Which electron has a larger uncertainty in momentum?

Consider the probability distribution $|\psi|^2$

$|\psi_A|^2 = \frac{1}{a\sqrt{2\pi}}e^{-x^2/2a^2}$

$|\psi_B|^2 = \frac{1}{a\sqrt{2\pi}}e^{-x^2/8a^2} = \frac{1}{a\sqrt{2\pi}}e^{-x^2/2(2a)^2}$



While part (b) doesn't require equations with typo to solve, we automatically give half credit for part (b).

Case B has a larger uncertainty Δx.

Uncertainty principle says $\Delta x\,\Delta p \geq \hbar/2$

Δp is smaller for case B.

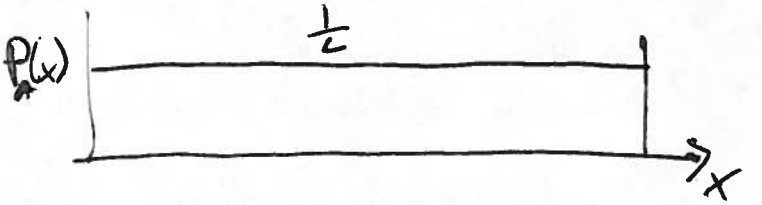
Δp is larger for case A.

(c) (10 points) Consider two possible wavefunctions for an electron: wavefunction A is $\psi_A(x,t=0)=\frac{1}{\sqrt{L}}e^{-jk_0x}$, and e wavefunction B is $\psi_B(x,t=0)=\frac{1}{\sqrt{a\sqrt{2\pi}}}e^{-x^2/4a^2}e^{-jk_0x}$. Why could wavefunction B be considered a better or more useful description for an electron than the wavefunction A? Give an explanation in text – however you may cite any relevant equations or figures.

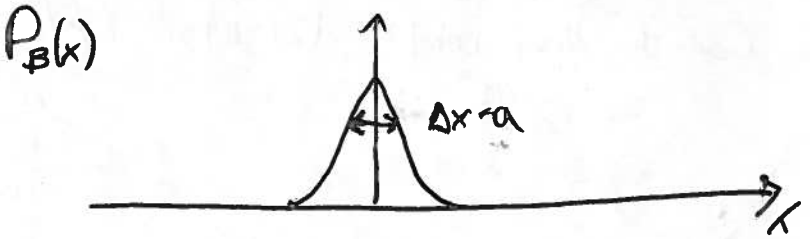
Case A is a plane wave which has infinite position uncertainty $\Delta x \to \infty$. This solution to Schrodinger's Eq. is "wave-like", but has no "particle-like" properties of localization.

Case B is a Gaussian wavepacket which has finite position uncertainty $\Delta x = a$. In other words, it is a better description of a localized particle like an electron, that still maintains wave properties. This represents wave-particle duality

$|\psi_A(x)|^2 = \frac{1}{L}$



$|\psi_B(x)|^2 = \frac{1}{a\sqrt{2\pi}}e^{-x^2/2a^2}$



3. Fermi Dirac distribution (30 points)

(a) (10 points) Consider non-degenerate GaAs with the Fermi level E_F fixed at 0.2 eV below the conduction band edge, i.e. E_C-E_F=0.2 eV. If the temperature increases from 200 K to 300 K, what is the ratio of the electron concentrations n(300K)/n(200K)?

$n = N_c e^{-(E_c-E_F)/kT}$ $T_1 = 300K$ $T_2 = 200K$

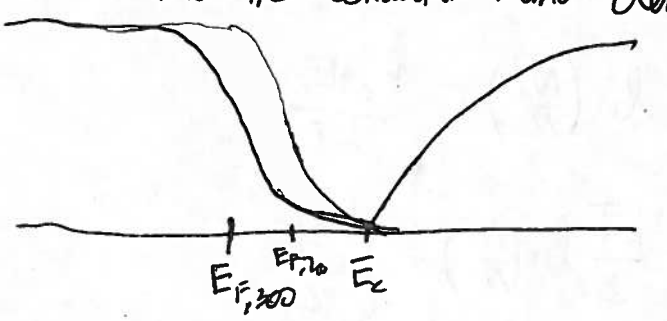
$$\frac{n(300)}{n(200)} = \frac{e^{-(E_c-E_F)/kT_1} N_c(300)}{e^{-(E_c-E_F)/kT_2} N_c(200)} = 1.84 \frac{e^{-0.2/0.026}}{e^{-0.2/0.0173}} = 1.84 \times 47.9 = 88$$

$N_c = 2\left(\frac{2\pi m^* kT}{h^2}\right)^{3/2}$ $\frac{N_c(T_1)}{N_c(T_2)} = \left(\frac{T_1}{T_2}\right)^{3/2} = 1.84$

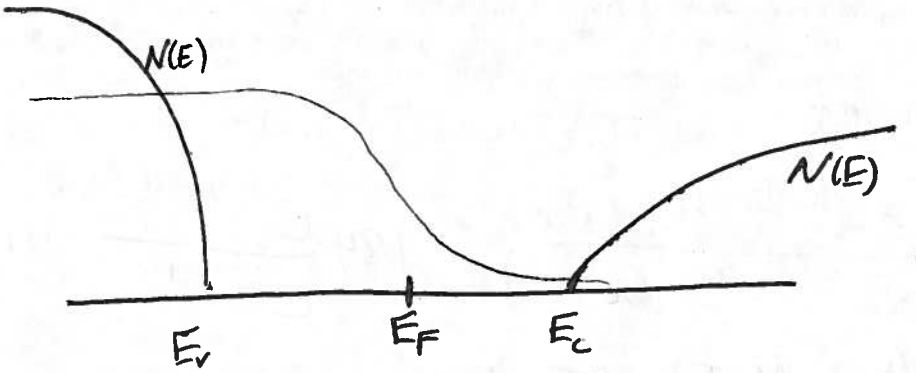
$$\boxed{\frac{n(300)}{n(200)} = 88}$$

(b) (10 points) Now assumed that the concentration n_0 of electrons in the conduction band is fixed at n_0=10^{13} cm^{-3}. If the temperature increases from 200K to 300K, will E_F increase, decrease, or stay the same? Why?

As temperature increases, the Fermi distribution tail extends further out into the conduction band density of states. Therefore, E_F must decrease in order to keep n_0 constant.



(c) (10 points) Now consider intrinsic GaAs in equilibrium at T=300 K. If the temperature increases to 400 K, will the intrinsic Fermi level energy increase, decrease, or stay the same? Why? You may use any combination of text, equations, and sketches to argue your answer.



The effective DOS is larger for the valance band than for the conduction band in GaAs because $m^*_p > m^*_n$. Therefore, in order to satisfy the intrinsic condition $n_i = p_i$, $E_F > \frac{E_c+E_v}{2}$, i.e. E_F is above the mid-gap energy.

$$n_i = p_i \Rightarrow N_c e^{-(E_c-E_F)/kT} = N_v e^{-(E_F-E_v)/kT}$$

$$N_c e^{-E_c/kT} e^{E_F/kT} = N_v e^{-E_F/kT} e^{E_v/kT}$$

$$e^{2E_F/kT} = \frac{N_v}{N_c} e^{(E_c+E_v)/kT}$$

$$\frac{2E_F}{kT} = \ln\left(\frac{N_v}{N_c}\right) + \frac{E_c+E_v}{kT}$$

$$E_F = \frac{kT}{2}\ln\left(\frac{N_v}{N_c}\right) + \frac{E_c+E_v}{2}$$

E_F will increase as the temperature increases.

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) \text{ for } 0<x<L,\ \psi(x)=0 \text{ otherwise} \quad , \ E_n=\frac{n^2\hbar^2\pi^2}{2mL^2}, \ n=1,2,...$$

Solution to 3D free particle in volume L^3: (with periodic boundary conditions)

$$\psi(x,y,z)=\frac{1}{L^{3/2}}e^{j\mathbf{k}\bullet\mathbf{r}} \ , \ E=\frac{\hbar^2k^2}{2m} \quad k_x=n_x\frac{2\pi}{L},k_y=n_y\frac{2\pi}{L},k_z=n_z\frac{2\pi}{L}, \ n_x,n_y,n_z=...-2,-1,0,1,2,...$$

Ohm's law: $\mathbf{J}=\sigma\mathbf{\mathcal{E}}$

Conductivity of free electron gas (i.e. metal): $\sigma=\frac{ne^2\bar{t}}{m}$ $\sigma=\rho^{-1}$

Conductivity of semiconductor: $\sigma=ne\mu_n+pe\mu_p$

Semiconductor electron/hole mobility $\mu_n=\frac{e\bar{t}_n}{m_n^*}$ $\mu_p=\frac{e\bar{t}_p}{m_p^*}$

3D free electron Density of States $N(E)=\frac{1}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}E^{1/2}$

Electron Effective mass $m^*=\frac{\hbar^2}{d^2E/dk^2}$

Fermi-Dirac distribution for electrons: $f(E)=\frac{1}{e^{(E-E_F)/kT}+1}$ and for holes: $f_h(E)=1-f(E)$

Equilibrium Carrier concentrations $n_0=\int_{E_C}^{\infty}f(E)N(E)dE$

Equilibrium Carrier concentrations in non-degenerate limit (E_C-$E_F \gg kT$ and E_F-$E_V \gg kT$).

$$n_0=N_Ce^{-(E_C-E_F)/kT} \ , \ p_0=N_Ve^{-(E_F-E_V)/kT}$$

$$N_C=2\left(\frac{2\pi m_n^*kT}{h^2}\right)^{3/2} \ , \ N_V=2\left(\frac{2\pi m_p^*kT}{h^2}\right)^{3/2}$$

$$n_i=\sqrt{n_0p_0}=\sqrt{N_CN_V}e^{-E_g/2kT}$$

Intrinsic Fermi Level $E_i=\frac{kT}{2}\ln\left(\frac{N_V}{N_C}\right)+\frac{E_V+E_C}{2}$

Einstein relation for diffusion coeff: $D=\frac{kT}{e}\mu$ Diffusion length $L=\sqrt{D\tau}$

Debye screening length (for n-type): $L_D=\sqrt{\frac{\epsilon kT}{e^2n_0}}$

Dielectric relaxation time $\tau_D=\frac{\epsilon}{\sigma}=\frac{\epsilon}{n_0e\mu_n}$ (n-type) $=\frac{\epsilon}{p_0e\mu_p}$ (p-type)

Fundamental constants

Planck's constant:	$h=6.63\times10^{-34}$ J s	$h=4.14\times10^{-15}$ eV s
	$\hbar=h/2\pi=1.06\times10^{-34}$ J s	$\hbar=h/2\pi=6.58\times10^{-16}$ eV s
Permittivity of free space	$\epsilon_0=8.85\times10^{-12}$ F/m	$\epsilon_0=8.85\times10^{-14}$ F/cm
Permeability of free space	$\mu_0=4\pi\times10^{-7}$ Ns^2/C^2	
Conversion from eV to J	1 eV=1.60×10^{-19} J	
Boltzmann's constant	$k=1.38\times10^{-23}$ J/K	$k=8.62\times10^{-5}$ eV/K
Bare electron mass	$m_0=9.11\times10^{-31}$ kg	
Speed of light	$c=2.998\text{x}10^{8}$ m/s	c=$2.998\text{x}10^{10}$ cm/s
Fundamental charge	$e=1.602\times10^{-19}$ C	

1 Å=10^{-10} m, 1 nm=10^{-9} m, 1 μm=10^{-6} m.

Material properties

Silicon

All parameters at room temp	Silicon	GaAs
Crystal Structure	Diamond	Zincblende
a	5.43 Å	5.65 Å
Mass density	2.33 g/cm^3	5.31 g/cm^3
ϵ_r	11.8	13.2
E_g	1.11 eV	1.43 eV
μ_n	1350 cm^2/V s	8500 cm^2/V s
μ_p	480 cm^2/V s	400 cm^2/V s
m^*_n	$0.26m_0$	$0.067m_0$
m^*_p	$0.49m_0$	$0.5m_0$
Effective DOS N_c	2.8×10^{19} cm^{-3}	4.7×10^{17} cm^{-3}
Effective DOS N_v	1.0×10^{19} cm^{-3}	7.0×10^{18} cm^{-3}
n_i	1.5×10^{10} cm^{-3}	2×10^{6} cm^{-3}

Useful equations

Electron momentum: $p=mv=\hbar k=h/\lambda$ Planck relation: $E=hf=\hbar\omega$

Time independent Schrodinger's Equation (1D): $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x)$

Expectation value for position $\langle x\rangle=\int_{-\infty}^{\infty}xP(x)dx=\int_{-\infty}^{\infty}x|\psi(x)|^2dx$. $\langle x^2\rangle=\int_{-\infty}^{\infty}x^2|\psi(x)|^2dx$.

Expectation value for momentum $\langle p\rangle=-j\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{\partial}{\partial x}\psi(x)kdx=\int_{-\infty}^{\infty}k|\psi(x)|^2dk$, Typo - incorrect formula

Uncertainty $\Delta x=\sqrt{\langle(x-\langle x\rangle)^2\rangle}=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$ $\Delta p_x=\sqrt{\langle(p_x-\langle p_x\rangle)^2\rangle}=\sqrt{\langle p_x^2\rangle-\langle p_x\rangle^2}$

Heisenberg uncertainty principle: $(\Delta x)(\Delta p_x)=\frac{\hbar}{2}$

Solution to 1D particle in box (infinite quantum well of width L with boundary condition $\psi(0)=\psi(L)=0$)

This page is left blank intentionally – use it for scrap paper.

This page is left blank intentionally – use it for scrap paper.