ECE-241A: Midterm Solution

Spring 2022

Problem 1 (Urns)

We pick consecutively, randomly and without putting them back, n balls from a urn. The urn contains r red balls and b blue balls, with $r+b \geq n$. Given that k of the n drawn balls are blue, what is the probability for the first drawn ball to be blue?

Solution

We assume that a unique index is associated to each ball: 1 to b for the blue balls, and $b+1$ to $b+r$ for the red balls. We can represent the random experiment of selection without putting back the balls as a vector of distinct integers $x_1, x_2, \ldots, x_n$, with $x_i \in \{1, \ldots, r+b\}$. Each vector represents an outcome, i.e., an element of Ω. They all have the same probability.
Given that we consider vectors with k blue balls, all the relevant outcomes have also same probability. Since the first drawn ball can be, with same probability, any of the n balls (among those there are k blue balls), the probability considered is $\frac{k}{n}$.

Alternatively, we can solve this problem using conditioning. Let B the event "the first ball is blue" and B_k the event "all the k balls have been drwan". Then

$$P(B|B_k) = \frac{P(B \cap B_k)}{P(B_k)} = \frac{P(B_k|B)P(B)}{P(B_k)}$$

with $P(B_k|B)$ the probability that a choice of $n-1$ balls from a urn containing r red balls and $b-1$ blue balls gives $k-1$ blue balls. Thus,

$$P(B_k|B) = \frac{\binom{b-1}{k-1}\binom{r}{n-k}}{\binom{r+b-1}{n-1}}.$$

With $P(B) = \frac{b}{r+b}$ and the hypergeometric probability $P(B_k) = \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{r+b}{n}}$, we get $P(B|B_k) = \frac{k}{n}$.

Problem 2

A communication system has n components: each of these components will be active, independently of the others, with probability p. The entire system can properly work only if half of its components are operational.

a. For which values of p is a 5-component system more often operational than a 3-component system?

b. In which situation will a system with $2k+1$ components be preferable to a system with $2k-1$ components?

Solution

a. The number of active components is a random variable of binomial law with parameter (n, p). Thus, the probability that a 5-component system works is

$$\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5$$

while the corresponding probability for a 3-component system is

$$\binom{3}{2}p^2(1-p) + p^3.$$

Consequently, the 5-component system is preferable if

$$10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$$

which can be reduced to

$$3(p-1)^2(2p-1) > 0$$

or $p > \frac{1}{2}$.

b. Let us show that a system with $2k+1$ is preferable if $p \geq \frac{1}{2}$. Let us consider a such system and denote by X the number of active components among the $2k-1$ first components. Then

$$P_{2k+1}[\{\text{System is active}\}] = P[X \geq k+1] + P[X = k](1-(1-p)^2) + P[X = k-1]p^2$$

since the $2k+1$ component system will be active in the following cases: $X \geq k+1$; $X = k$ and at least one of the two remaining components is active; $X = k-1$ and the last two component are both active. Since $P_{2k-1}[\{\text{System is active}\}] = P[X \geq k] = P[X = k] + P[X \geq k+1]$, we get

$$\begin{aligned} P_{2k+1}[\{\text{System is active}\}] - P_{2k-1}[\{\text{System is active}\}] &= P[X = k-1]p^2 - (1-p)^2 P[X = k] \\ &= \binom{2k-1}{k-1}p^{k-1}(1-p)^k p^2 - (1-p)^2\binom{2k-1}{k}p^k(1-p)^{k-1} \\ &= \binom{2k-1}{k-1}p^k(1-p)^k(p-(1-p)) \\ &> 0 \Longleftrightarrow p > \frac{1}{2} \end{aligned}$$

by noticing that $\binom{2k-1}{k-1} = \binom{2k-1}{k}$.

Problem 3

Let X and Y be two independent random variables of law $U[0,1]$. What is the conditional law of the random variable $(X-Y)^+$ knowing Y?

Solution

Let $Z = (X-Y)^+$. Note that X and Y are independent. When conditioning on $Y = y \in [0,1]$, $X \sim [0,1]$. We have

$$\Pr[X \leq y | Y = y] = y \quad \text{and} \quad \Pr[X \leq y+z | Y = y] = y + z \text{ for } 0 \leq z \leq 1-y.$$

Moreover, $\Pr[X \le y+z] = 1$ for $z \ge 1-y$. Hence, we have the conditional c.d.f for $Z|Y$ as

$$F_{Z|Y}[z|y] = \Pr[(X-y)^+ \le z | Y=y] = \begin{cases} 0, & \text{for } z < 0 \\ \Pr[X \le y | Y=y] = y, & \text{for } z = 0 \\ \Pr[X \le y+z | Y=y] = y+z, & \text{for } 0 \le z < 1-y, \\ 1, & \text{for } z \ge 1-y. \end{cases}$$

Take a derivative of $F_{Z|Y}[z|y]$ w.r.t. z, we have the p.d.f. as

$$f_{Z|Y}[z|y] = \mathbb{I}_{z\in(0,1-y]} + \delta(z)\cdot y$$

where $\delta(z)$ is the Dirac delta function with $\forall z' \ne 0 : \delta(z') = 0$ and $\int_{-\infty}^{\infty} \delta(z)dz = 1$.

Problem 4

Consider $X_1, X_2, \ldots, X_n$ a sequence of random variables with mean 0 and variance 1. Let $S_n = \sum_i X_i/n$.

a. Is it possible to show that $|S_n|$ is upper bounded 0.1 with probability at least 0.9 for very large n (e.g., $n > 100000000$)? If yes, prove it. If no, give a counter example.

b. Suppose for any $i \ne j$, X_i and X_j are independent (but any three of them may not be independent), i.e., these random variables are pairwisely independent. Under this condition, do Problem (a) again. (**Hint:** Look at the variance of S_n and try to use Chebyshev's inequality. Try to expand the formula of $\mathrm{Var}(S_n)$, use linearity of expectation and $\mathbb{E}(X_iX_j) = \mathbb{E}(X_i)\mathbb{E}(X_j)$ for $i \ne j$.)

Solution

a. No. Let us consider the case where

$$X_1 = \begin{cases} 1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2, \end{cases}$$

and $X_1 = X_2 = \cdots = X_n$. Each X_i has mean 0 and variance 1. In this case, you can notice that the X_i are not i.i.d, but the statement of the question authorizes it. We will have $S_n = 1$ with probability 1.

b. Since $\mathbb{E}[X_i] = 0$ for any i, then $\mathbb{E}[S_n] = 0$. We also have

$$Var(S_n) = \frac{1}{n^2} Var(\sum_i X_i) = \frac{1}{n^2}\sum_{i,j}(X_i, X_j) = \frac{1}{n}\sum_i Var(X_i) = \frac{1}{n}.$$

Using Chebyshev, we have

$$\mathbb{P}[|S_n - 0| \le \varepsilon] = \mathbb{P}[|S_n - \mathbb{E}[S_n]| \le \varepsilon] \ge 1 - \frac{1}{\varepsilon^2 \cdot n} > 0.9.$$