part I (4)
$\vec{E}$ formula (1)
$\vec{r}$ (1)
$\rho_s\, a\, d\alpha$ (1)   (4) Symmetry argument

(I) formula
(1) Limit
(1) Ans

part II $V = -\int \vec{E}\cdot d\vec{l}$ (3)

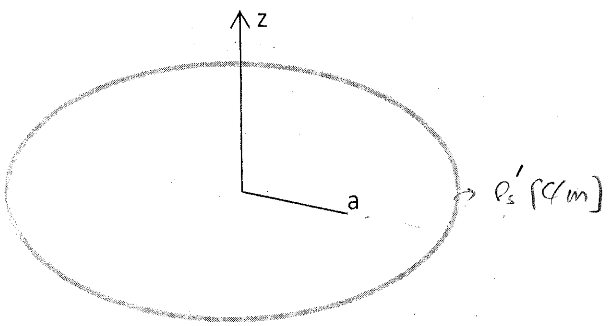
part III $E = \frac{kQ}{r}$ (3) (1) formula
(1) setup
(1) Ans

EE1 Midterm

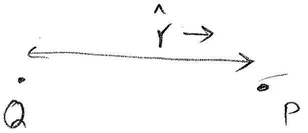
Fall, 2010

Name: Solution

#1. (35pts) For the case of a ring of uniform line charge density ρ_s shown in the following figure, the radius of the ring is a. find (1) electric field anywhere in Z-axis from Coulomb's law (4) (2) the potential anywhere in Z-axis through path integration of the electric field (3) (3) the potential anywhere in Z-axis through charge-potential integration (3)



(1) $\vec{E} = \frac{KQ}{r^2}\hat{a}_r$ (I) ⇒ Formula



(2)

(1) $\vec{E}$ formula $= \frac{kQ\,\hat{a}_r}{r^2}$

(2) $V = -\int \vec{E}\cdot d\vec{l}$ (2)

(3) $V = \frac{kQ}{r}$ (2)

$\vec{r} = -a\hat{a}_\rho + z\hat{a}_z$
$\Rightarrow |\vec{r}| = \sqrt{a^2+z^2}$ (I) ⇒ unit vector

$$\Rightarrow d\vec{E} = K\frac{\overbrace{\rho_s\, a\, d\alpha}^{(I)\ \to\ \text{Charge Integration}}\,[-a\hat{a}_\rho + z\hat{a}_z]}{(a^2+z^2)^{3/2}}$$

Due to symmetry,
No $\vec{E}$ along $\hat{a}_\rho$ direction

(4) $\rho_s\, d\alpha$ (2)
(5) $\hat{r}$ (2)

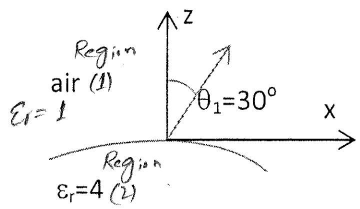
$$\Rightarrow \vec{E} = \int_{(\alpha=0)}^{2\pi}\frac{K\rho_s\, a z\,\hat{a}_z}{(a^2+z^2)^{3/2}}\,d\alpha = \frac{\rho_s\, a z\,(2\pi)\,\hat{a}_z}{4\pi\varepsilon_0\,(a^2+z^2)^{3/2}} = \frac{\rho_s\, a z}{2\varepsilon_0\,(a^2+z^2)^{3/2}}\hat{a}_z = \vec{E}(z)$$ (I)

(2) (I)
$$V(z) = -\int_{\infty}^{z}\vec{E}\cdot d\vec{l} = -\int_{\infty}^{z}\frac{\rho_s\, a z}{2\varepsilon_0(a^2+z^2)^{3/2}}\,dz = -\frac{\rho_s a}{4\varepsilon_0}\int_{\infty}^{z}(a^2+z^2)^{-3/2}(2z)\,dz$$

$$= +\frac{\rho_s a}{4\varepsilon_0}\frac{[a^2+z^2]^{-1/2}}{-1/2}\Bigg|_{\infty}^{z} = \frac{\rho_s a}{2\varepsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{a^2+\infty^2}}\right] = \frac{\rho_s\, a}{2\varepsilon_0\sqrt{a^2+z^2}}.$$

(3) $V(r) = \frac{KQ}{r} \Rightarrow V(z) = K\int_0^{2\pi}\frac{\rho_s\, a\, d\alpha}{(a^2+z^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_s\, a\,(2\pi)}{\sqrt{a^2+z^2}} = \frac{\rho_s\, a}{2\varepsilon_0\sqrt{a^2+z^2}}$

#2. (30pts) For a boundary surface between air and a dielectric with dielectric constant of 4, the electric field in the air right above the surface is measured to be 30 degree from the normal direction of the surface (assumed to be z-axis) and with a magnitude of 1. Write the electric field and electric flux density in the dielectric right below the surface in vector form. If the dielectric has a conductivity σ=0.1, what would be the current density excited by the field?



Region - I ⇒

$$\vec{E_1} = 0.5\hat{a_x} + 0.866\hat{a_z}.$$

$$\vec{D_1} = \varepsilon\vec{E_1} = \underbrace{\varepsilon_0\, 0.5\hat{a_x}}_{\text{tangential component}} + \underbrace{\varepsilon_0\, 0.866\hat{a_z}}_{\text{Normal component}}$$

Boundary Conditions

$$\vec{E_{1t}} = \vec{E_{2t}}$$
$$\vec{D_{1N}} = \vec{D_{2N}} \quad (3)$$

Region - II

$$\vec{E_{2t}} = \vec{E_{1t}} = 0.5\hat{a_x}$$
$$\vec{D_{2N}} = \vec{D_{1N}} = \varepsilon_0(0.866)\hat{a_z}$$

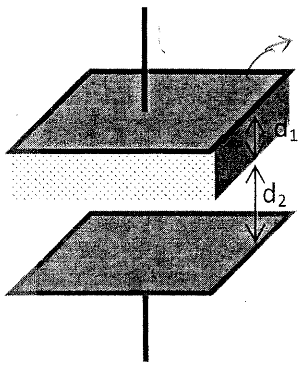
$$\varepsilon_0(4)\vec{E_{2N}} = \varepsilon_0(0.866)\hat{a_z} \Rightarrow \vec{E_{2N}} = 0.2165\hat{a_z}$$

$$\vec{E_2} = 0.5\hat{a_x} + 0.2165\hat{a_z} \quad (3)$$

$$\vec{D_2} = \varepsilon\vec{E_2} = \underbrace{\varepsilon_0\, 2\hat{a_x}}_{\text{tangential Component}} + \underbrace{\varepsilon_0\, 0.866\hat{a_z}}_{\text{Normal component}} \quad (2)$$

$$\vec{J_2} = \sigma\vec{E_2} \rightarrow (1)$$
$$= (0.1)[0.5\hat{a_x} + 0.2165\hat{a_z}] \quad (2)$$
$$= 0.05\hat{a_x} + 0.02165\hat{a_z}. \rightarrow (1)$$

#3. (35pts) For a parallel plate capacitor with a layer of dielectric (ε_r=4) inserted shown in the following figure. The area of the plate is S and the thickness of the dielectric layer and the air layer are d_1 and d_2 respectively. (1) Derive its capacitance from Gauss's law. (2) If the capacitor is standalone, one takes the dielectric away from it, will the energy stored in the capacitor increase or reduce? (3) If the capacitor is connected to a voltage source with constant voltage V_0, one takes the dielectric away from it again, will the energy stored in the capacitor increase or reduce?



let's say charge $= +Q$. $\Rightarrow$ Charge Density $= \frac{Q}{S}$ Gauss law ①

$D_{N_1} = \frac{Q}{S} \Rightarrow E_{IN} = \frac{D_{IN}}{\varepsilon_1} = \frac{Q}{\varepsilon_1 S}$.

$D_{N_2} = \frac{Q}{S} \Rightarrow E_{2N} = \frac{D_{2N}}{\varepsilon_2} = \frac{Q}{\varepsilon_2 S}$. ②

$\Rightarrow V_1 = E_{IN} d_1 = \frac{Q d_1}{\varepsilon_1 S}$

$\Rightarrow V_2 = E_{2N} d_2 = \frac{Q d_2}{\varepsilon_2 S}$.

$$\Rightarrow \boxed{C = \frac{Q}{V_1 + V_2} = \frac{1}{\frac{d_1}{\varepsilon_1 S} + \frac{d_2}{\varepsilon_2 S}}} \quad ①$$

(2) Cap $\Rightarrow$ stand along. Take Dielectric away from It. will energy stored in cap Increase or reduce.

$E_{stored} = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C}$.

$Q \Rightarrow$ remains constant.	$Q \Rightarrow$ remains constant ③
C reduce	V_2 : Same
$E \uparrow$	V_1 : Increase
	$V_1 + V_2 \Rightarrow$ Increase
	$E \uparrow$

(3) Cap out $\Rightarrow$ while plates connected to a source.

$\Rightarrow$ V across plates fixed $\Rightarrow$ As Dielectric moved away ~~E_{IN} tends to~~

Q decreases ③