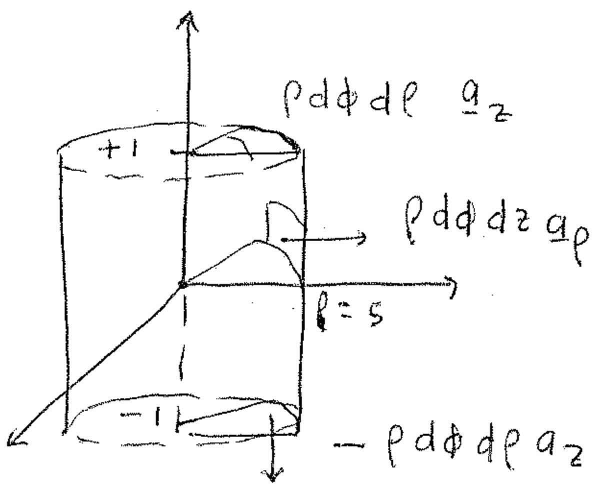
Solution to 1

2 Divergence theorem is $\oint \underline{D} \cdot d\underline{s} = \int (\underline{\nabla} \cdot \underline{D})\, dv$

$$\underline{D} = 2\rho z^2\, \underline{a}_\rho + \rho \cos^2\phi\, a_z.$$

Choose Gaussian Surface a cylinder

3 

LHS of divergence theorem

3 $$\int_0^{2\pi}\int_{-1}^{+1} (2\rho z^2)\, \underline{a}_\rho \cdot \rho\, d\phi\, dz\ \underline{a}_\rho \qquad \text{(side)}$$

3 $$\int_0^{2\pi}\int_0^{5} \cancel{(\rho \cos^2\phi)}\, \underline{a}_z \cdot \rho\, d\phi\, d\rho\ \underline{a}_z. \qquad \text{(top)}$$

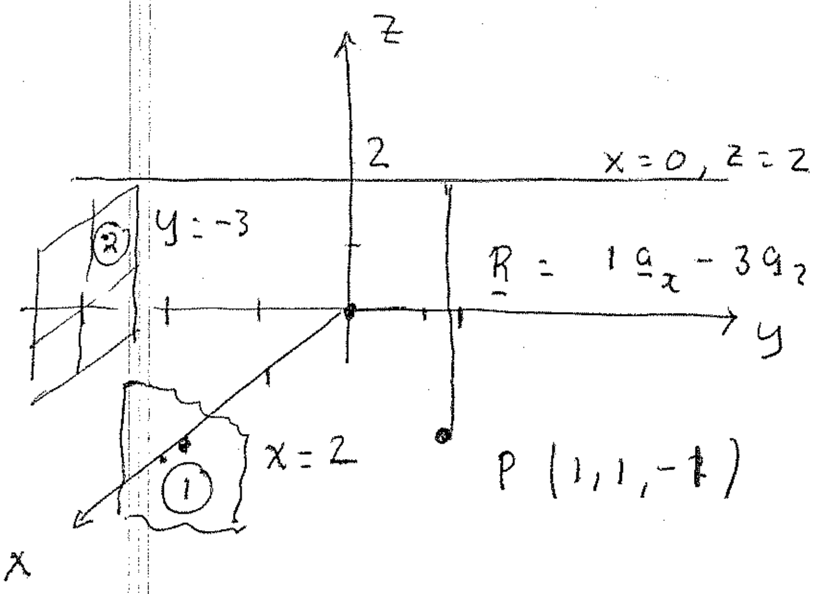
3 $$-\int_0^{2\pi}\int_0^{5} \cancel{(\rho \cos^2\phi)}\, a_z \cdot \rho\, d\phi\, d\rho\ (\underline{a}_z) \qquad \text{(bottom)}$$

1 $$2\rho^2 (2\pi)\left(\frac{z^3}{3}\right)_{-1}^{1}\Bigg|_{\rho=5} = 2(25)(2\pi)\left(\frac{2}{3}\right)$$

1 $$= \frac{200\pi}{3} = 209.44\ C$$

25 points

Solution to 2



$\underline{R} = 1\underline{a}_x - 3\underline{a}_z$ so $\rho = \sqrt{10}$ and

$$\underline{a}_\rho = \frac{1\underline{a}_x - 3\underline{a}_z}{\sqrt{10}}$$

We use principle of linear superposition

5 $\underline{E}(p) = \underline{E}_1(p)\Big|_{\text{sheet charge at } x=2} + \underline{E}_2(p)\Big|_{\text{sheet charge at } y=-3} + \underline{E}_3(p)\Big|_{\text{line charge } x=0,\ z=2}$

5 $$\underline{E}_1(p) = \frac{\rho_{s1}}{2\varepsilon_0}(-\underline{a}_x) = -\frac{10\times10^{-9}}{2\times8.85\times10^{-12}}\underline{a}_x = -565\,\underline{a}_x$$

5 $$\underline{E}_2(p) = \frac{\rho_{s1}}{2\varepsilon_0}(+\underline{a}_y) = \frac{15\times10^{-9}}{2\times8.85\times10^{-12}}\underline{a}_y = 848\,\underline{a}_y$$

5 $$\underline{E}_3(p) = \frac{\rho_L}{2\pi\varepsilon_0\rho}\underline{a}_\rho = \frac{\cancel{10}\,\cancel{\pi}\times10^{-9}}{2\cancel{\pi}\times8.85\times10^{-12}\times\sqrt{10}}\left(\frac{\underline{a}_x - 3\underline{a}_z}{\sqrt{\cancel{10}}}\right)$$

$$= 56.5\,(\underline{a}_x - 3\underline{a}_z)$$

5 $\underline{E}(p) = -565\,\underline{a}_x + 848\,\underline{a}_y + 56.5\,\underline{a}_x - 169.5\,\underline{a}_z$ V/m

$= -508.5\,\underline{a}_x + 848\,\underline{a}_y - 169.5\,\underline{a}_z$ V/m

25 points

Solution to 3.

Electric field of a point charge is given by

$$\underline{E} = \frac{Q}{4\pi\varepsilon_0 r^2} a_r$$

and the work done or potential difference

5 $$W = V_{ba} = -\int_{\text{initial } a}^{\text{final } b} \underline{E} \cdot d\underline{L} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \cancel{\frac{1}{r_a}}\right]$$

let $r_a \to \infty$

5 For $Q_1 = -4\times10^{-6}$ C $\quad r_{b_1} = \sqrt{(1-2)^2 + (0+1)^2 + (1-3)^2}$

$$= \sqrt{6}$$

5 For $Q_2 = 5\times10^{-6}$ C $\quad r_{b_2} = \sqrt{(1)^2 + (0-4)^2 + (1+2)^2}$

$$= \sqrt{26}$$

Use superposition

5 $$V_{total}(1,0,1) = \frac{10^{-6}}{4\pi\times8.85\times10^{-12}}\left[-\frac{4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]$$

$$= 9\times10^3(-1.63 + 0.99)$$

5 $$\simeq -5.87 \text{ kV}$$

(2) 4 Coulombs law $\underline{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\, \underline{a}_r$ (N)

(2) Gauss law $\oint_S \underline{D}\cdot d\underline{s} = Q_{enclosed}$ (C)

(2) Maxwell's equation $\underline{\nabla}\cdot\underline{D} = \rho_v$ (C/m^3)

(2) Potential Difference $V_{ab} = -\int_a^b \underline{E}\cdot d\underline{L}$ (V)

(2) Electric flux density $\underline{D} = \varepsilon_0 \underline{E}$ (C/m^2)

(2) Volume element (cyl co-ord) $dv = \rho\, d\phi\, d\rho\, dz$.

(6) Surface elements (-ll-)

$$d\underline{s} = \rho\, d\phi\, dz\, \underline{a}_\rho$$
$$= \rho\, d\rho\, d\phi\, \underline{a}_z$$
$$= d\rho\, dz\, \underline{a}_\phi.$$

$$\nabla \times A = \cancel{\underline{A}\times\underline{B}} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} =$$

(2)

$$= \underbrace{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)}_{1} a_x + \underbrace{\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)}_{2} a_y + \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{3} a_z$$

(5)

$$\underline{\nabla}\cdot(\underline{\nabla}\times\underline{A}) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(3)$$

$$= \cancel{\frac{\partial^2 A_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_y}{\partial x \partial z}} - \cancel{\frac{\partial^2 A_z}{\partial y \partial x}} - \frac{\partial^2 A_x}{\partial y \partial z} + \cancel{\frac{\partial^2 A_y}{\partial z \partial x}} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$= 0$ always $\equiv 0$ no matter what $\underline{A}$ is.

Solutions.

5 a) $$(\underline{\nabla} \cdot \underline{a})\,\underline{M} = 3\underline{M} = 6zy\,\underline{a}_x + 3xy^2\,\underline{a}_y + 3x^2yz\,\underline{a}_z.$$

5 b) $$(\underline{a} \cdot \underline{\nabla})\,\underline{M} = x\frac{\partial \underline{M}}{\partial x} + y\frac{\partial \underline{M}}{\partial y} + z\frac{\partial \underline{M}}{\partial z}$$

$$= x\left(y^2\,\underline{a}_y + 2xyz\,\underline{a}_z\right) + y\left(2z\,\underline{a}_x + 2xy\,\underline{a}_y + x^2z\,\underline{a}_z\right)$$

$$+ z\left(2y\,\underline{a}_x + x^2y\,\underline{a}_z\right)$$

$$= 4yz\,\underline{a}_x + 3xy^2\,\underline{a}_y + 4x^2yz\,\underline{a}_z.$$

5 c) $$\underline{\nabla} \cdot \underline{a}\,(\underline{a} \cdot \underline{M}) = 3\left(2xyz + xy^3 + x^2yz^2\right)$$

$$= 6xyz + 3xy^3 + 3x^2yz^2$$

5 d) $$(\underline{a} \cdot \underline{\nabla})\,a^2 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right)$$

$$= 2\left(x^2 + y^2 + z^2\right) = 2a^2.$$

(2) 4 Coulomb's law $\underline{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\, \underline{a}_r$ (N)

(2) Gauss law $\oint_S \underline{D} \cdot d\underline{s} = Q_{enclosed}$ (C)

(2) Maxwell's equation $\underline{\nabla} \cdot \underline{D} = \rho_v$ (C/m^3)

(2) Potential Difference $V_{ab} = -\int_a^b \underline{E} \cdot d\underline{L}$ (V)

(2) Electric flux density $\underline{D} = \varepsilon_0 \underline{E}$ (C/m^2)

(2) Volume element (cyl co-ord) $dv = \rho\, d\phi\, d\rho\, dz$.

(6) Surface elements (-"-)

$$d\underline{s} = \rho\, d\phi\, dz\, \underline{a}_\rho$$
$$= \rho\, d\rho\, d\phi\, \underline{a}_z$$
$$= d\rho\, dz\, \underline{a}_\phi.$$

$$\underline{A} \times \underline{B} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} =$$

(2)
$$= \underbrace{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)}_{1} a_x + \underbrace{\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right)}_{2} a_y + \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)}_{3} a_z$$

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(3)$$

(5)
$$= \cancel{\frac{\partial^2 A_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_y}{\partial x \partial z}} - \cancel{\frac{\partial^2 A_z}{\partial y \partial x}} - \frac{\partial^2 A_x}{\partial y \partial z} + \cancel{\frac{\partial^2 A_y}{\partial z \partial x}} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$= 0$ always $\equiv 0$ no matter what $\underline{A}$ is.