EE 1 Midterm Fall 2007 Name: Solution

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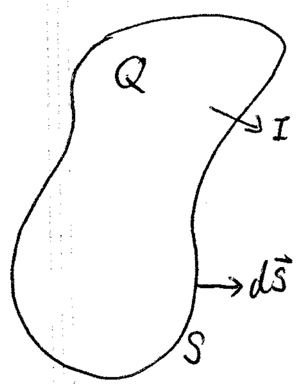
(7) | a Starting from the definition of the current density J derive the continuity equation.



$$I = \oint_S \vec{J}\cdot d\vec{s} \quad (1)$$, where $\vec{J}$ is current density (A/m²)

Left side : $I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_v dv = \int_V \left[-\frac{d\rho_v}{dt}\right] dv.$ (2)

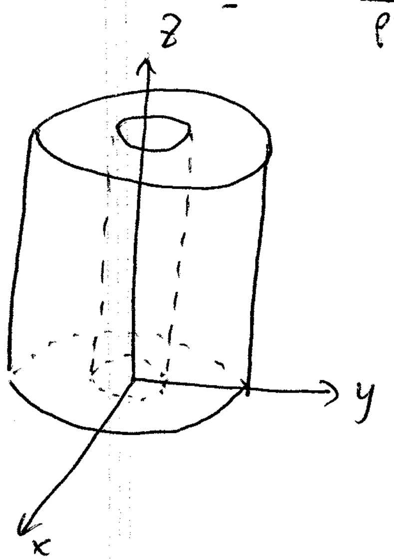
Right side : $\oint_S \vec{J}\cdot d\vec{s} = \int_V (\nabla\cdot\vec{J})\, dv.$ (divergence theorem) (2)

$$\Rightarrow \int_V \left[-\frac{d\rho_v}{dt}\right] dv = \int_V (\nabla\cdot\vec{J})\, dv$$

remove vol. int. $\Rightarrow -\frac{d\rho_v}{dt} = \nabla\cdot\vec{J}.$ (2)

(8) | b Find the total outward current crossing a closed cylindrical surface defined by $\rho_1 = 0.01$ m and $\rho_2 = 0.4$ m and $z = 0$ and $z = 0.2$ m if

$$\underline{J} = \frac{25}{\rho^2}\,\underline{a}_\rho - \frac{20}{\rho^3}\,\underline{a}_z \quad A/m^2$$



$$I = \oint_S \vec{J}\cdot d\vec{s} \quad (1)$$

$$= \int_{top} \vec{J}\cdot d\vec{s} + \int_{bottom} \vec{J}\cdot d\vec{s} + \int_{outer} \vec{J}\cdot d\vec{s} + \int_{inner} \vec{J}\cdot d\vec{s} \quad (1)$$

$$= \cancel{\int_0^{2\pi}\int_{0.01}^{0.4}\left(-\frac{20}{\rho^3}\right)\rho\, d\rho\, d\phi} \ (1) + \cancel{\int_0^{2\pi}\int_{0.01}^{0.4}\left(-\frac{20}{\rho^3}\right)(-\rho\, d\rho\, d\phi)} \ (1)$$

$$+ \int_0^{0.2}\int_0^{2\pi}\frac{25}{\rho^2}\rho\, d\phi\, dz\Big|_{\rho=0.4} \ (1) + \int_0^{0.2}\int_0^{2\pi}\frac{25}{\rho^2}(-\rho\, d\phi\, dz)\Big|_{\rho=0.01} \ (1)$$

$$= 25\pi - 1000\pi$$

$$= -975\pi \ (A) \quad (2)$$

(5) 2 a Starting from Gauss' law derive Maxwell's $\nabla \cdot D$ equation

Gauss' Law : $Q_{encl} = \oint_S \vec{D} \cdot d\vec{s}$ (1)

Left side : $Q_{encl} = \int_v \rho_v dv$ (1)

Right side : $\oint_S \vec{D} \cdot d\vec{s} = \int_v (\nabla \cdot \vec{D}) dv$ (divergence theorem) (1)

$\Rightarrow \int_v \rho_v dv = \int_v (\nabla \cdot \vec{D}) dv$ (1)

remove vol. int. $\Rightarrow \rho_v = \nabla \cdot \vec{D}$ (1)

(10) 2 b If the electric flux density $D = 6\rho \sin\frac{\phi}{2}\, a_\rho + 1.5\rho \cos\frac{\phi}{2}\, a_z$ C/m^2 evaluate the total charge in a region bounded by

$\rho = 2$

$0 \leq \phi \leq 2\pi$

$0 \leq z \leq 5$

Hint $\nabla \cdot D = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$.

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method 1 : $\rho_v = \nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \cdot 6\rho \sin\phi/2) + \frac{1}{\rho}\frac{\partial 0}{\partial \phi} + \frac{\partial}{\partial z}(1.5\rho\cos\phi/2)$ (2)

$= 12 \sin\phi/2 + 0 + 0$ (2)

$= 12 \sin\phi/2$ (C/m^3) (1)

$Q = \int_v \rho_v dv = \int_0^5 \int_0^{2\pi} \int_0^2 12 \sin\phi/2 \cdot \rho\, d\rho\, d\phi\, dz$. (2)

$= 12 \times 5 \times (-2\cos\phi/2)\Big|_0^{2\pi} \times \frac{1}{2}\rho^2\Big|_0^2$ (2)

$= 480$ (C) (1)

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method 2 : $Q = \oint_S \vec{D} \cdot d\vec{s} = \int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} + \int_{side} \vec{D} \cdot d\vec{s}$ (2)

$= \cancel{\int_0^{2\pi}\int_0^2 1.5\rho\cos\phi/2 \cdot \rho\, d\rho\, d\phi}$ (2) $+ \cancel{\int_0^{2\pi}\int_0^2 1.5\rho\cos\phi/2 \times (-\rho\, d\rho\, d\phi)}$ (2)

$+ \int_0^5 \int_0^{2\pi} 6\rho \sin\phi/2 \cdot \rho\, d\phi\, dz \Big|_{\rho=2}$ (2)

$= 6\rho^2\Big|_{\rho=2} \times 5 \times (-2\cos\phi/2)\Big|_0^{2\pi}$ (1)

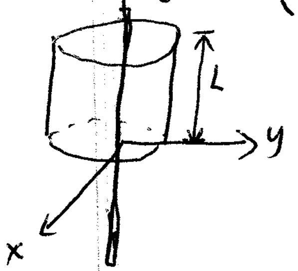
$= 480$ (C). (1)

Derive using Gauss' law the electric field produced by an infinite line charge

8 3a ~~An~~ infinite line charge density $\rho_L = 10\ \text{C/m}$ lies ~~[illegible]~~ along the z axis.

Find the electric field at a point P (1,1,1)

$\varepsilon_0 = \left(\frac{1}{36\pi} \times 10^{-9}\right)$ F/m



Because of the symmetry of this problem

$\vec{E} = E_\rho(\rho)\,\vec{a_\rho}$ 1

Choose a closed cylindrical surface with height L,

$Q_{encl} = \oint_S \vec{D}\cdot d\vec{s} = \oint_S \varepsilon_0 \vec{E}\cdot d\vec{s}$ 1

$Q_{encl} = \rho_L \cdot L$ 1

$\oint_S \varepsilon_0 \vec{E}\cdot d\vec{s} = \cancel{\int_{top} \varepsilon_0 \vec{E}\cdot d\vec{s}}^{\,0} + \cancel{\int_{bottom} \varepsilon_0 \vec{E}\cdot d\vec{s}}^{\,0} + \int_{side} \varepsilon_0 \vec{E}\cdot d\vec{s}$

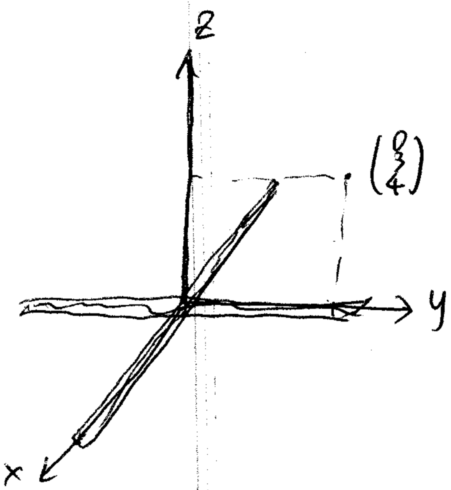
$= \int_{side} \varepsilon_0 E_\rho(\rho)\,\vec{a_\rho}\cdot ds\,\vec{a_\rho}$ 1 $= \int_0^L\int_0^{2\pi} \varepsilon_0 E_\rho(\rho)\,\rho\, d\phi\, dz$

$= \varepsilon_0 E_\rho(\rho)\cdot\rho\cdot 2\pi\cdot L$ 1

$\Rightarrow E_\rho(\rho) = \frac{\rho_L \cdot L}{2\pi\varepsilon_0 \rho \cdot L} = \frac{\rho_L}{2\pi\varepsilon_0\rho}$, $\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a_\rho}$ 1

At P (1,1,1), $\rho = \sqrt{1^2+1^2} = \sqrt{2}$, $\vec{E_P} = \frac{10}{2\pi\times\varepsilon_0\times\sqrt{2}}\vec{a_\rho} = 90\sqrt{2}\times 10^9\,\vec{a_\rho}$ (V/m) 2

7 3b Now calculate E at (0,3,4) if infinite uniform line charges of 5 nC/m lie along the X and the Y axis.



For x-axis line charge,

$\rho = \sqrt{3^2+4^2} = 5$, $\vec{a_\rho} = \frac{1}{5}(3\vec{a_y} + 4\vec{a_z})$ 2

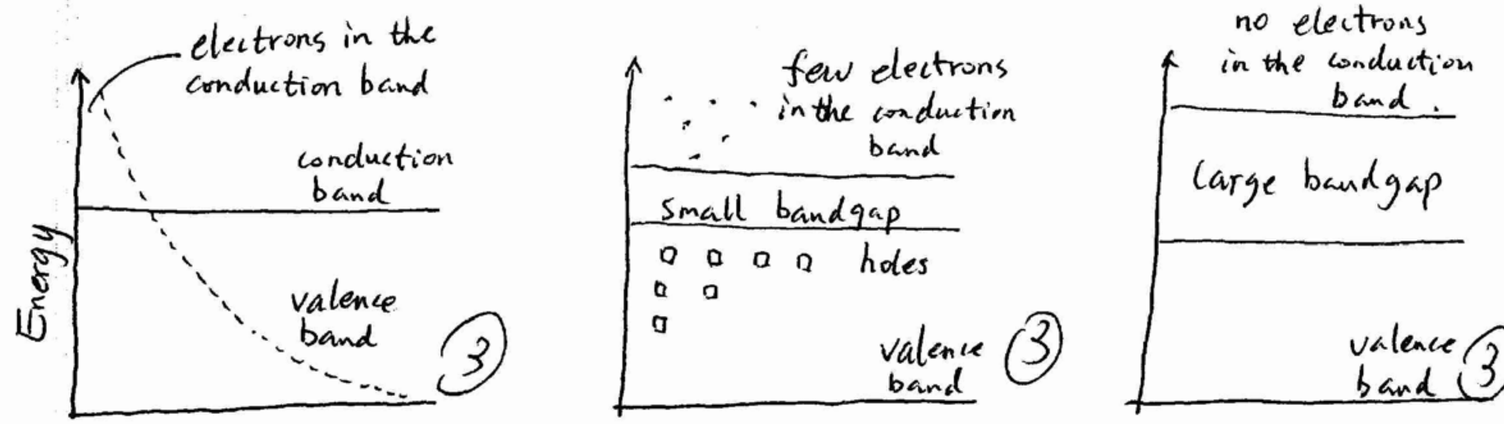
For y-axis line charge,

$\rho = 4$, $\vec{a_\rho} = \vec{a_z}$ 2

$\Rightarrow \vec{E} = \frac{5\times 10^{-9}}{2\pi\times\varepsilon_0\times 5}\times\frac{3\vec{a_y}+4\vec{a_z}}{5} + \frac{5\times 10^{-9}}{2\pi\times\varepsilon_0\times 4}\times\vec{a_z}$ 2

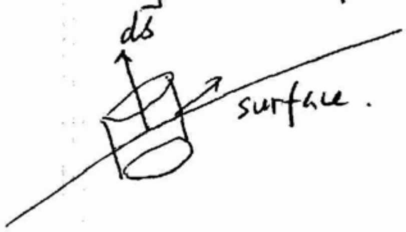
$= 10.8\,\vec{a_y} + 36.9\,\vec{a_z}$ (V/m) 1

9 4 a Explain with the aid of a diagram the main difference between a conductor, a semiconductor and a dielectric.



conducter | semi-conductor | Dielectric

3 b What are the boundary conditions for E_t and E_n at the interface of a conductor and a vacuum?



$$E_t = 0. \quad (1)$$

$$D_n = \rho_s \Rightarrow E_n = \frac{D_n}{\varepsilon_0} = \frac{\rho_s}{\varepsilon_0} \quad (2)$$

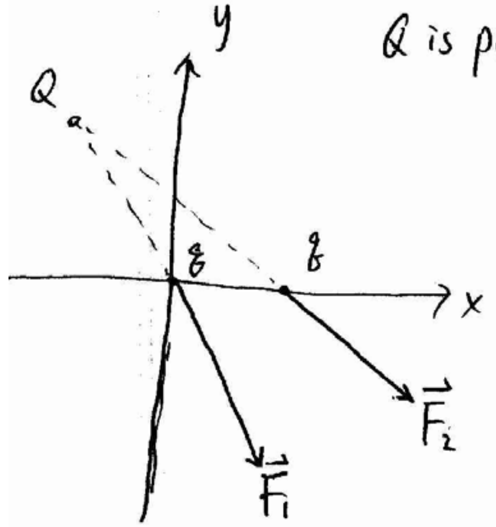
2 c What happens when and electric field is applied across a conductor? / a dielectric?

Conductor: The charges move under the influence of the applied field, but because of continual collisions, they aquire a drift velocity. (1)

Dielectric: The bound charges cannot move but are displaced, resulting in the formation of dipoles and leading to the polarization of the medium. (1)

1 d What happens to the conductivity of a conductor as temperature is reduced? Conductivity increases as temperature is reduced. (1)

Q5. A positive charge Q is placed at (x, y, z). When a unit charge is placed at the origin the force on it is observed to be in the direction $0.5\,a_x - 0.5\sqrt{3}\,a_y$. When the unit charge is moved along the x axis by 1 unit the force is in the direction $0.6\,a_x - 0.8\,a_y$. Find the co-ordinates (x, y, z) of charge Q?



Q is placed at $\vec{r}' = x\,\vec{a}_x + y\,\vec{a}_y + z\,\vec{a}_y$.

$$\vec{r} = a\,\vec{a}_x + b\,\vec{a}_y + c\,\vec{a}_z$$

$$\vec{R} = \vec{r} - \vec{r}' = (a-x)\,\vec{a}_x + (b-y)\,\vec{a}_y + (c-z)\,\vec{a}_z$$

$$\vec{F} = \frac{Q \cdot q\,[(a-x)\vec{a}_x + (b-y)\vec{a}_y + (c-z)\vec{a}_z]}{4\pi\varepsilon_0\,[(a-x)^2 + (b-y)^2 + (c-z)^2]^{3/2}} \quad (3)$$

At origin, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

$$\vec{F}_1 = \frac{Q \cdot q\,[-x\vec{a}_x - y\vec{a}_y - z\vec{a}_z]}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{3/2}} = 0.5\,\vec{a}_x - 0.5\sqrt{3}\,\vec{a}_y \quad (2)$$

At $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

$$\vec{F}_2 = \frac{Q \cdot q\,[(1-x)\vec{a}_x - y\vec{a}_y - z\vec{a}_z]}{4\pi\varepsilon_0\,[(1-x)^2 + y^2 + z^2]^{3/2}} = 0.6\,\vec{a}_x - 0.8\,\vec{a}_y \quad (2)$$

No z-component $\Rightarrow z = 0$. (1)

$$\frac{F_{1x}}{F_{1y}} = \frac{x}{y} = \frac{0.5}{-0.5\sqrt{3}} \Rightarrow y = -\sqrt{3}\,x \quad (2)$$

$$\frac{F_{2x}}{F_{2y}} = \frac{1-x}{-y} = \frac{1-x}{\sqrt{3}\,x} = \frac{0.6}{-0.8} \Rightarrow x^2 + 2.91x - 1.45 = 0 \quad (2)$$

$x_1 = 0.434$ (1)

$x_2 = -3.344$

Because it is observed $x<0$, $y>0$, $x_2 = -3.344$ is the only solution. (1)

$y = -\sqrt{3}\,x = 5.792$.

$\Rightarrow x = -3.344,\quad y = 5.792,\quad z = 0$. (1)