EE 141 – Final

06/12/08
Duration: 3 hours

The final is closed book and closed lecture notes. No calculators.
You can use a single page of handwritten notes.
Please carefully justify all your answers.

Problem 1: (30 points) The linearized equations describing the vertical of motion of a hot-air balloon are given by:

$$\begin{aligned} \tau_1 \dot{T} &= -T + u \\ \tau_2 \ddot{z} + \dot{z} &= aT + w \end{aligned}$$

were T represents the deviation of the hot-air temperature from the equilibrium temperature, z represents the altitude of the balloon, u represents the deviation of the burner heating rate from the equilibrium rate, and w is the wind speed. In what follows we will assume that $w = 0$ and to simplify the computations the parameters τ_1, τ_2, and a will assume the following unrealistic values:

$$\tau_1 = 0.1 \qquad \tau_2 = 0.2 \qquad a = 10$$

1. (6 points) Compute the transfer function from the input u to the balloon's altitude.

   The Laplace transform of the first and second differential equations gives:

   $$\frac{T}{U} = \frac{1}{\tau_1 s + 1} \qquad \frac{Z}{T} = \frac{a}{s(\tau_2 s + 1)},$$

   assuming zero initial conditions and $w = 0$. Therefore:

   $$\frac{Z}{U} = \frac{Z}{T}\frac{T}{U} = \frac{10}{s(0.1s + 1)(0.2s + 1)}$$

2. (12 points) Design a compensator for a unity-feedback loop so that the steady-state error to a parabola is 0.2.

Let D be transfer function of the compensator and let $G = Z/U$. For a unity-feedback configuration we have:

$$\frac{Z}{R} = \frac{DG}{1+DG} \qquad \frac{E}{R} = \frac{R-Z}{R} = \frac{1+DG-DG}{1+DG} = \frac{1}{1+DG}$$

We want to design D so that:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\frac{1}{1+DG}\frac{1}{s^3} = \lim_{s\to 0} \frac{1}{1+DG}\frac{1}{s^2} = 0.2$$

We first note that:

$$\lim_{s\to 0} \frac{1}{1+DG}\frac{1}{s^2} = \lim_{s\to 0} \frac{(0.1s+1)(0.2s+1)}{s^2(0.1s+1)(0.2s+1)+10sD(s)}$$

If $D(s)$ is a simple gain, the above limit is not finite, therefore we pick $D(s) = K/s$ to obtain:

$$\lim_{s\to 0} \frac{1}{1+DG}\frac{1}{s^2} = \frac{1}{10K},$$

and we conclude that $K = 1/2$.

3. (12 points) Synthesize compensators C and D, as in Figure 1, so that the closed loop transfer function becomes:

$$\frac{5}{s^3+15s^2+50s+50}$$

We write the plant as $B_g(s)/A_g(s)$ and the desired closed-loop as $B_h(s)/A_h(s)$. The first step is to decompose B_g, B_h and A_d as $B_g = B_g^+ B_g^-$, $B_h = B_g^+ B_h'$ and $A_d = B_g^- A_d'$ where:

$$B_g^+ = 10 \quad B_g^- = 1 \quad B_h' = \frac{1}{2}$$

The first equation to be solved is:

$$A_d' A_g + B_d B_g^+ = A_h$$

If we chose $A_d' = a$ and $B_d = b$ we obtain:

$$as(0.1s+1)(0.2s+1) + b10 = s^3 + 15s^2 + 50s + 50$$

The solution is:

$$a = 50 \qquad b = 5$$

Compensator D is thus given by:

$$D = \frac{B_d}{A_d} = \frac{B_d}{B_g^- A_d'} = \frac{5}{50} = \frac{1}{10}$$

Compensator C is given by:

$$C = \frac{B_h'}{B_d} = \frac{1/2}{5} = \frac{1}{10}$$

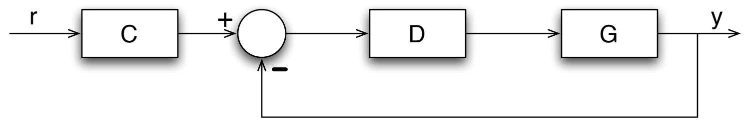

Figure 1: Closed-loop system for Problem 1.

Problem 2: 35 points

1. (10 points) Sketch the Bode plot for the system described by the transfer function:

$$H(s) = \frac{30(s+1)}{s^2 + 8s + 25}$$

knowing that $\log_{10}(30/25) \approx 0.1$.

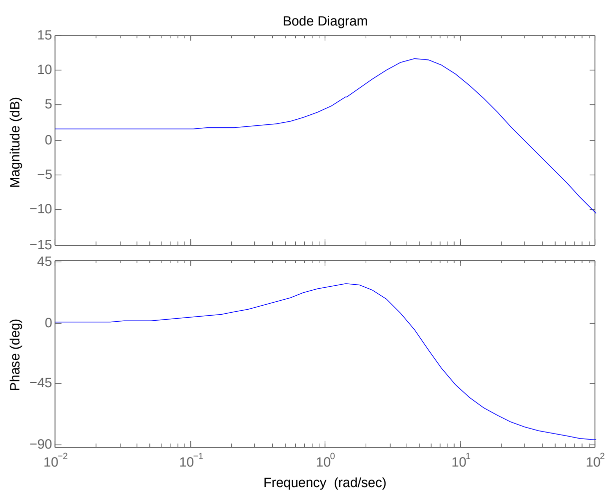

Figure 2: Bode plot for Problem 2.

2. (5 points) How would your Bode plot change if the zero $s = -1$ is replaced by the zero $s = 1$?

   The magnitude would remain unaltered, but the phase evolution of the zero would be a change from -180^o at low frequencies to -270^o at high frequencis with -225^o at $\omega = 1$. This can be easily seen by noting that for $\omega << 1$, $(j\omega - 1) \approx -1$ and for $\omega >> 1$, $j\omega - 1 \approx j\omega$.

3. (10 points) Based on your plot, compute the phase margin and the gain margin. What can you conclude regarding stability of this system for various values of a controller gain K in a unity-feedback loop? Justify your answer.

The gain margin is infinite since the phase never reaches -180^o. This means that this system is stable for all positive values of a control gain K. The phase margin is about 100^o.

4. (10 points) Use the Routh test to verify your previous answer.

For a unity-feedback loop the closed-loop transfer function is:

$$\frac{KG}{1+KG} = \frac{K30(s+1)}{s^2+8s+25+K30(s+1)}$$

The Routh table for the above characteristic polynomial is:

$$\begin{array}{c|cc} 2 & 1 & 30K+25 \\ 1 & 30K+8 & 0 \\ 0 & 30K+25 & \end{array}$$

Stability is ensured if all the elements in the first column (to the right of the line) are positive. This requires $K > -8/30$ and $K > -25/30$ which is clearly satisfied for all positive K. The Routh test thus confirms the previous answer.

Problem 3: **(35 points)** The Nyquist plot of the system represented by the transfer function:

$$H(s) = \frac{(s-1)(s+2)}{(s+1)(s-4)(s-5)}$$

is shown in Figure 3. The plot crosses the horizontal axis approximately at the points -0.125, -0.1 and 0.

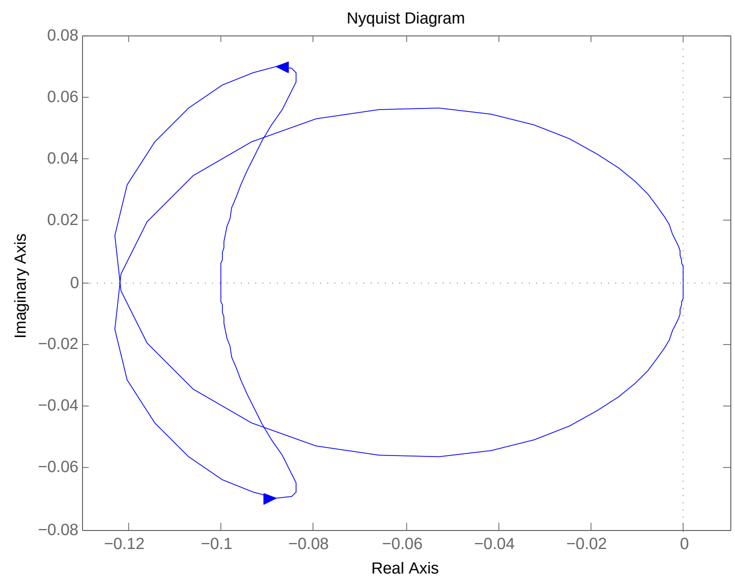

Figure 3: Nyquist plot for Problem 3.

1. (14 points) Determine the number of unstable closed-loop poles for all positive values of a control gain K.

   The plant has 2 poles on the right side of the complex plane and the number Z of closed-loop poles is given by $N = Z - 2 \Leftrightarrow Z = N + 2$ where N is the number of times that a particular point on the negative horizontal axis is encircled by the Nyquist diagram. Inspecting the diagram we observe that all the points smaller than -0.125 are encircled 0 times, all the points between -0.125 and -0.1 are encircled -2 times and all the points between -0.1 and 0 are encircled -1 times. Since a point $-p$ on the negative horizontal axis corresponds to a control gain $-1/K$ we conclude that for gains between 0 and 8 there are $0 + 2 = 2$ unstable closed-loop poles, for control gains between 8 and 10 there are $-2 + 2 = 0$ unstable closed-loop poles, and for control gains larger than 10 there are $-1 + 2 = 1$ unstable closed-loop poles.

2. (14 points) Verify the answer to the above question by sketching a root-locus. Justify your answer.

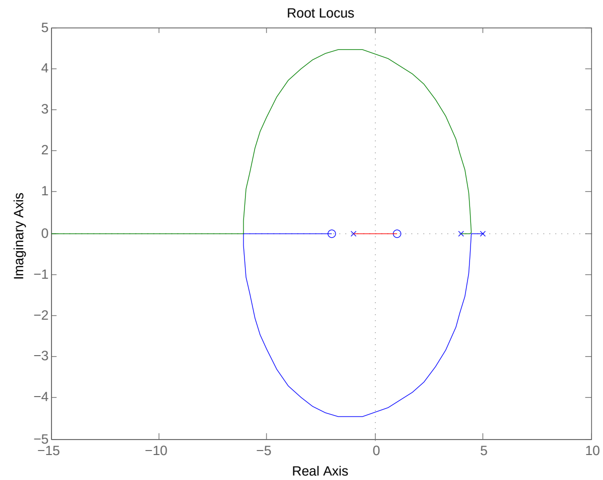

Figure 4: Root locus plot for Problem 3.

Observing the root locus we see that we start with two unstable poles (gain between 0 and 8 as computed in the previous question), by increasing the gain these poles become stable (gain between 8 and 10 as computed in the previous question), and by further increasing the gain, the third pole that was stable becomes unstable (gain larger than 10 as computed in the previous question). The root-locus thus confirms the analysis done on the Nyquist diagram.

3. (7 points) Which Bode plot in Figure 5 corresponds to the Nyquist plot in Figure 3?

   The third plot is excluded since it starts with a phase of 0^o and non-neglible amplitude. However, the Nyquist plot does not contain points on the right side of the complex plane. For the same reason the first plot is excluded. It contains points with non-neglibible amplitude for phases between 90^o and 0^o. The Bode plot thus correspond to the second plot.

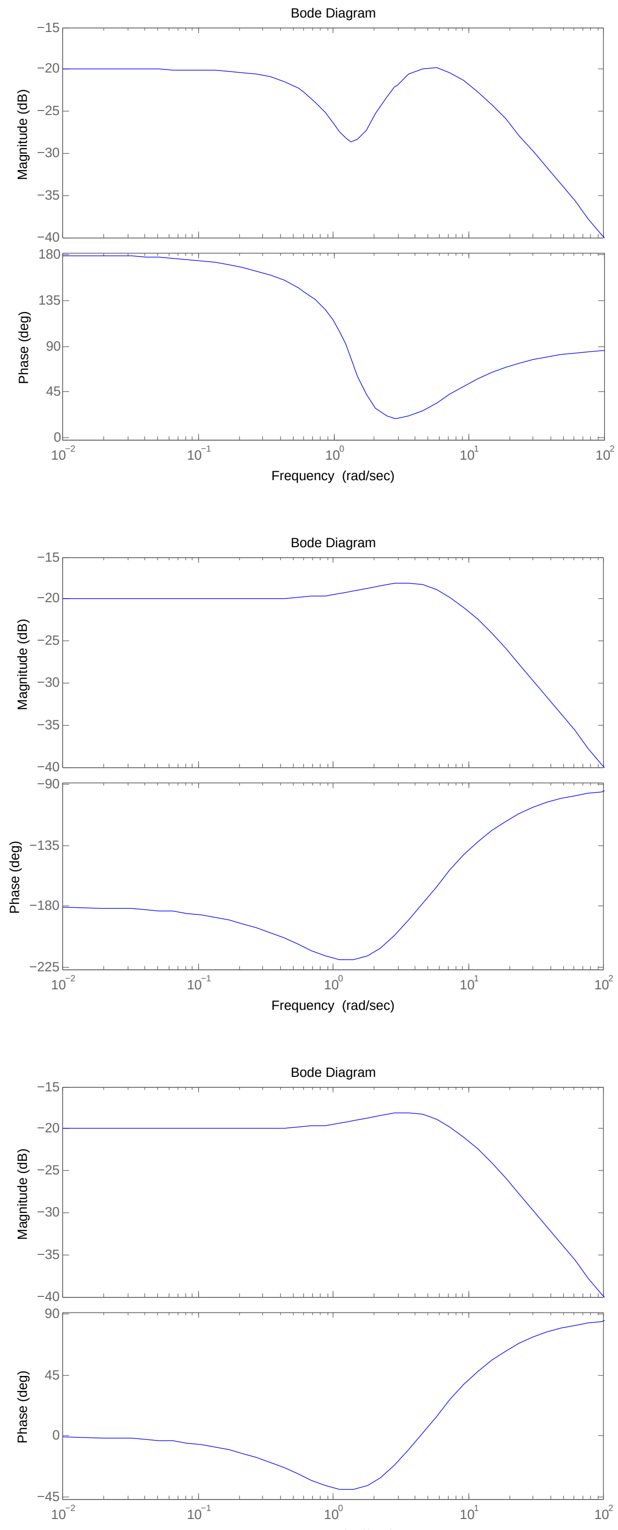

Figure 5: Bode plots for Problem 3.