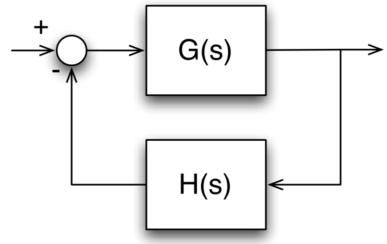


Figure 1: Feedback interconnection for Problem 1.

Consider the following linear differential equation:

$$\frac{d}{dt}x_1 = -x_1 + 2x_2 \tag{1}$$
$$\frac{d}{dt}x_2 = x_2 + u \tag{2}$$

1. Assuming that $x_1(0) = 0$ and $x_2(0) = 0$, what is the evolution of x_1, in the time domain, when a step is applied as input at $t = 0$?

   We first apply the Laplace transform to the differential equations (1) and (2), using the fact that $x_1(0) = 0$ and $x_2(0) = 0$, to obtain:

   $$sX_1(s) = -X_1(s) + 2X_2(s) \tag{3}$$
   $$sX_2(s) = X_2(s) + U(s) \tag{4}$$
   $$\tag{5}$$

   Eliminating X_2 in the above equations leads to:

   $$X_1(s) = \frac{2}{(s-1)(s+1)}U(s)$$

   If the input is a step applied at $t = 0$ we have $U(s) = \frac{1}{s}$ so that:

   $$X_1(s) = \frac{2}{(s-1)(s+1)s} = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s+1}$$

   The coefficients in the partial fraction expansion are given by:

   $$a = X_1(s)s|_{s=0} = -2 \quad b = X_1(s)(s-1)|_{s=1} = 1 \quad c = X_1(s)(s+1)|_{s=-1} = 1$$

   Taking the inverse Laplace transform of $X_1(s)$ we finally obtain:

   $$\mathcal{L}^{-1}\{X_1(s)\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = -2u(t) + e^t u(t) + e^{-t}u(t)$$

2. Assume that we place a controller with transfer function $H(s) = K_D s + K_P$ in a feedback loop with the system defined by the differential equations (1) and (2), as depicted in Figure 1. Design K_D and K_P so that the rise time of the closed-loop system is not greater than 0.9 seconds and the damping ratio is 1.

   The closed-loop transfer function is given by:

$$\frac{G}{1+GH} = \frac{2}{s^2 + 2K_D s + 2K_P - 1}$$

   Writing the denominator in the form $s^2 + 2\zeta\omega_n s + \omega_n^2$ we conclude:

$$\omega_n = \sqrt{2K_P - 1} \tag{6}$$

$$\zeta = \frac{K_D}{2\sqrt{2K_P - 1}} \tag{7}$$

   Using the approximation $\omega_n \geq \frac{t_r}{0.9}$, valid for second order systems without zeros, we conclude that $t_r \leq 0.9$ implies $\omega_n \geq 2$. We thus pick $\omega_n = 2$ and obtain $K_P = 2.5$ from (6). Using now (7) and the requirement $\zeta = 1$ we obtain $K_D = 2$.

3. For your design, compute the overshoot and the settling time.

   Since the damping ratio is one, there is no overshoot. For a second order system without zeros, the settling time is approximately given by $\frac{4.6}{\sigma}$. We obtain σ by factorizing $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 4s + 4$ as $(s+2)(s+2)$. Therefore, $\sigma = 2$ and $t_s \approx 2.6$.

4. Sketch the step response.

   The step response can be sketched knowing the rise time, the settling time, the overshoot, and the DC gain. All these ingredients have been computed before except for the DC gain that is given by $\lim_{s\to 0} \frac{2}{s^2+4s+4} = \frac{2}{4} = \frac{1}{2}$.

5. Would the closed-loop system be able to track a ramp?

   If we define the error $e = r - y$ as the difference between the reference signal r and the output y we obtain:

$$\frac{E}{R} = \frac{R-Y}{R} = 1 - \frac{Y}{R} = 1 - \frac{2}{s^2+4s+4} = \frac{s^2+4s+2}{s^2+4s+4}$$

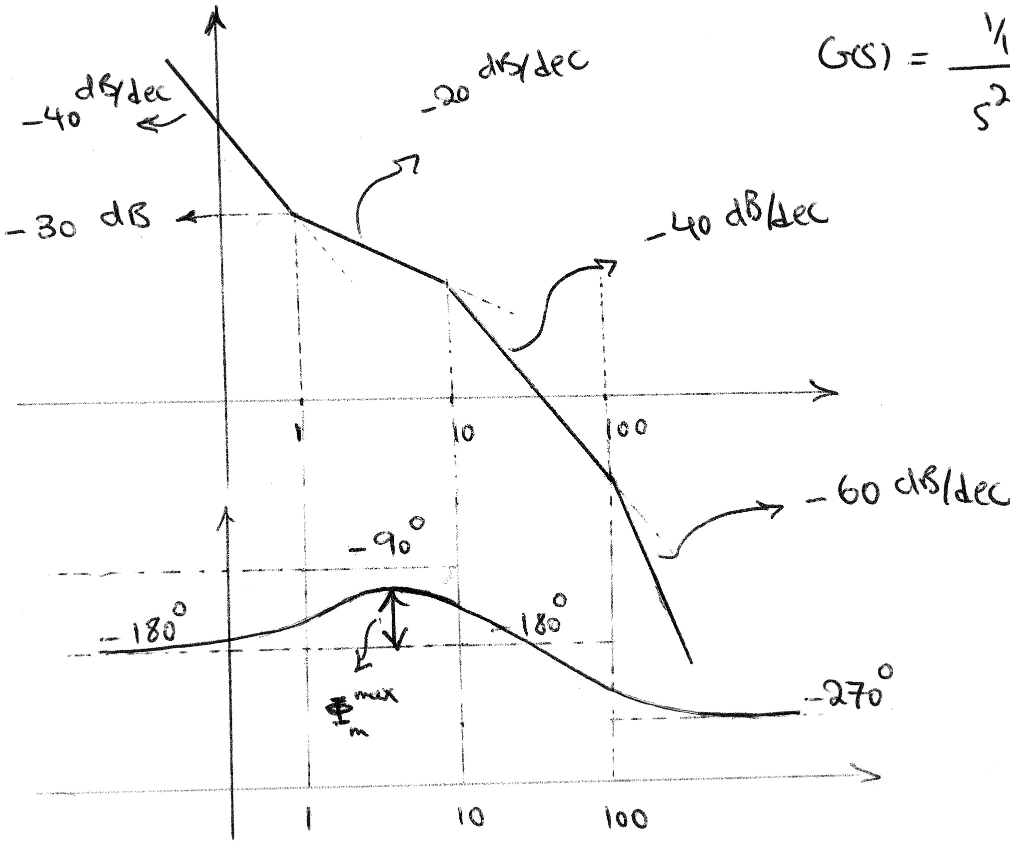
   As the poles of E/R are stable, we can compute the steady state error to a ramp input as:

$$\lim_{s\to 0} s\frac{E}{R}\frac{1}{s^2} = \lim_{s\to 0} s\frac{s^2+4s+2}{s^2+4s+4}\frac{1}{s^2} = \lim_{s\to 0} \frac{s^2+4s+2}{s(s^2+4s+4)} = \infty$$

   The system cannot track a ramp since the steady state error is not zero.

Problem 2

$$G(s) = \frac{\frac{1}{1000}(s+1)}{s^2\left(1+\frac{s}{10}\right)\left(1+\frac{s}{100}\right)}$$



PM ≃ 0°

$PM_{max} \simeq 45°$

Problem 3

1. $x_1 \triangleq x$, $x_2 \triangleq \dot{x} = \dot{x}_1$ $\Rightarrow$ $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -kx_1 + x_2^2 + \frac{1}{2}x_2^3 + u \end{cases}$

$\dot{x}_1 = 0 \Rightarrow x_2^\circ = 0 \Rightarrow \boxed{\dot{x}^\circ = 0}$

$\dot{x}_2 = 0 \Rightarrow u^\circ = kx_1^\circ \Rightarrow \boxed{u^\circ = kx^\circ}$

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2. $\ddot{x} + kx = (\dot{x})^2 + \frac{1}{2}(\dot{x})^3 + u$

@ $\begin{cases} \dot{x}^\circ = 0 \\ u^\circ = kx^\circ \end{cases} \rightarrow \ddot{x} + kx = (0)^2 + \frac{1}{2}(0)^3 + kx$

$\Rightarrow \ddot{x} = 0 \Rightarrow x(t) = mt$

$\Rightarrow$ None of the equilibrium pairs corresponds to a sinusoidal motion.

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3. $u = -\frac{1}{2}\dot{x}^3 - (\dot{x})^2 + v \Rightarrow \ddot{x} + kx = v$

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4. $\boxed{v_1 = -k_1 x} \Rightarrow \ddot{x} + (k + k_1)x = 0$

$\omega_0 = 2\pi \times 261.63 \Rightarrow k + k_1 = \omega_0^2 \Rightarrow \boxed{k_1 = \omega_0^2 - k}$

If k is not exactly known, a different note will be generated.

5

$$\boxed{v_2 = -k_1 x - k_2 \dot{x}} \Rightarrow \ddot{x} + k_2 \dot{x} + (k + k_1) x = 0$$

$$\Rightarrow k + k_1 = \omega_o^2 \Rightarrow \boxed{k_1 = \omega_o^2 - k}$$

$$k_2 = 2\xi \omega_o , \quad 0 < \xi < 1$$

$$\Downarrow$$

$$\boxed{k_2 = 2\xi \omega_n}$$

If k is not exactly known, a different note with decreasing intensity will be generated.

6 For v_2: Yes, Since the controller v_2 includes x - dynamic, $\dot{x}(t)$.

For v_1: No, Since the controller v_1 does not include any derivative of $x(t)$.