ECE 131A
Probability and Statistics
Instructor: Lara Dolecek

Winter 2022 Exam 2 Solutions
Feb 16, 2022
Maximum score: 100 points

You have 2 hours to submit your work **directly on Gradescope under the Exam_2 submission link**.
Please read and carefully follow all the instructions.

Instructions

- The exam is accessible from 12:00 am PST on February 16th to 11:59 pm PST on February 16th. Once you open the exam, you will have 2 hours to upload your work (therefore open the exam at least 2 hours before the closing time).
- This exam is open book, open notes. You are allowed to consult your own class notes (homework, discussion, lecture notes, textbook). You are not allowed to consult with each other or solicit external sources for help (e.g., an online forum).
- For each question, start a new sheet of paper. Therefore, the number of pages of your scan should be at least the number of questions. It is ok to write multiple parts of a question on one sheet. Properly erase or cross out any scratch work that is not part of the answer.
- Please submit your exam through the corresponding submission link on Gradescope.
- Make sure to include your **full name** and **UID** in your submitted file.
- Make sure to **show all your work**. Unjustified answers will be at a risk of losing points.
- **Policy on the Academic Integrity**
  "During this exam, you are **disallowed** to contact with a fellow student or with anyone outside the class who can offer a solution e.g., web forum."
  Please write the following statement on the first page of your answer sheet. You will **lose 10 points** if we can not find this statement.

I ___*YourName*___ with UID ____ have read and understood the policy on academic integrity.

0. Please read the Policy on the Academic Integrity on the first page of this exam and follow the instructions.

1. (15 pts) Suppose X is a Gaussian RV with mean 2 and variance 4. Express $P(-1 < X < 3)$ in terms of the Q-functions.
   Solution:

$$
\begin{aligned}
P(-1 < X < 3) &= P(\frac{-1-2}{2} < \frac{X-2}{2} < \frac{3-2}{2}) \\
&= P(-\frac{3}{2} < Z < \frac{1}{2}) \\
&= Q(-\frac{3}{2}) - Q(\frac{1}{2})
\end{aligned}
$$

2. (15 pts) X is a discrete random variable that counts the number of failures when flipping a coin with success probability p before a success comes up. We can relate X to a geometric random variable A with parameter p by writing $X = A - 1$. Compute the characteristic function of X and find its first moment (i.e. $\mathbf{E}[X]$) using the characteristic function.

   Solution: The pmf for this random variable is

$$P(X = k) = (1-p)^k p$$

   for $k \geq 0$ and is zero otherwise.

   As such,

$$\begin{aligned}
\Phi_X(w) = \mathbb{E}[e^{jwX}] &= \sum_{k=0}^{\infty} e^{jwk} P(X = k) \\
&= p \sum_{k=0}^{\infty} e^{jwk} (1-p)^k \\
&= p \sum_{k=0}^{\infty} (e^{jw}(1-p))^k \\
&= \frac{p}{1-(1-p)e^{jw}}.
\end{aligned}$$

$$\frac{d}{d\omega}\Phi_X(\omega) = \frac{jp(1-p)e^{j\omega}}{(1-(1-p)e^{j\omega})^2}$$

$$\begin{aligned}
\mathbf{E}[X] &= \frac{1}{j}\frac{d}{d\omega}\Phi_X(\omega)|_{\omega=0} \\
&= \frac{p(1-p)}{(1-(1-p))^2} \\
&= \frac{1}{p} - 1
\end{aligned}$$

   This is the result that we expect since $\mathbb{E}[X] = \mathbb{E}[A] - 1 = \frac{1}{p} - 1$.

3. (15 pts) Suppose X is a Gaussian RV with mean 0 and variance 25. Let RV Y be defined as follows:

$$Y = \begin{cases} X, & \text{if } X \geq 2, \\ 4 - X, & \text{if } X < 2. \end{cases}$$

Compute the pdf of Y. You may leave your answer in terms of the $Q()$ for $\Phi()$ functions where $\Phi(x) = 1 - Q(x)$.

Solution:

First, note that Y can only take values $y \geq 2$. Thus, for $y \geq 2$,

$$\begin{aligned} P(Y \leq y) &= P(Y \leq y, X \geq 2) + P(Y \leq y, X < 2) \\ &= P(X \leq y, X \geq 2) + P(4 - X \leq y, X < 2) \\ &= P(2 \leq X \leq y) + P(4 - y \leq X < 2) \\ &= P(4 - y \leq X \leq y) = P(\frac{4-y}{5} \leq \frac{X}{5} \leq \frac{y}{5}) \\ &= \Phi(\frac{y}{5}) - \Phi(\frac{4-y}{5}) \end{aligned}$$

where $\Phi(\cdot)$ is the cdf for the standard normal distribution.

Thus, for $y \geq 2$,

$$\begin{aligned} f_Y(y) &= \frac{1}{5}\Phi'(\frac{y}{5}) + \frac{1}{5}\Phi'(\frac{4-y}{5}) \\ &= \frac{1}{5\sqrt{2\pi}}(e^{-\frac{y^2}{50}} + e^{-\frac{(4-y)^2}{50}}) \end{aligned}$$

and $f_Y(y) = 0$ for $y < 2$.

4. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) A discrete random variable has jump discontinuities in its cumulative distribution function.
Solution: TRUE

(b) The probability density function of a continuous random variable is always an increasing function in its argument.
Solution: FALSE

(c) For a random variable X with mean μ and variance σ, Chebyshev's inqequality states that

$$P(|X - \mu| \geq a) \leq \frac{\sigma}{a^2}$$

Solution: TRUE

(d) The following function $f(x)$ is a valid pdf:

$$f(x) = \begin{cases} \frac{2x}{7}, & -3 \leq x \leq 4, \\ 0, & \text{otherwise}, \end{cases}$$

Solution: FALSE

(e) The Markov inequality holds for both discrete and continuous random variables.
Solution: TRUE

5. (20 pts) Let $X = \sqrt{U}$. Find the CDF and PDF of X when

(a) U is a uniform random variable in $[0, 1]$.

(b) U is an exponential random variable with parameter 4.

Solution:

(a) U is a uniform random variable in $[0, 1]$
The CDF is shown as the following:

$$F_X(x) = P[X \leq x] = P[\sqrt{U} \leq x] = P[U \leq x^2]$$

Where $0 \leq x^2 \leq 1$.
Since $P[U \leq k] = k$ for $0 \leq k \leq 1$, then we have:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Therefore we have the PDF as:
$f_X(x) = \frac{d}{dx}F_X(x) = 2x$ for $0 \leq x \leq 1$, and $f_X(x) = 0$ elsewhere.

(b) U is an exponential random variable with parameter 4.
We have $F_U(u) = 1 - e^{-4u}$, $u \geq 0$. Now,

$$F_X(x) = P[X \leq x] = P[\sqrt{U} \leq x] = P[U \leq x^2]$$

We have,

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-4x^2}, & 0 \leq x \leq \infty \end{cases}$$

$f_X(x) = \frac{d}{dx}F_X(x) = 8xe^{-4x^2}$ for $0 \leq x \leq \infty$, and $f_X(x) = 0$ elsewhere.

6. (20 pts) Suppose an RV X has the following pdf:

$$f(x) = \begin{cases} Cxe^{-x} & x \geq 0 \\ 0 & else \end{cases}$$

(a) Find the constant C.
Solution:

Note that e^{-x} is the pdf for an exponential RV with parameter 1 resulting in mean 1 and variance 1. Thus, $\int_0^\infty xe^{-x}dx = 1$, $\int_0^\infty x^2e^{-x}dx = 2$.
Using $\int_{-\infty}^\infty f(x)dx = 1$ and the fact that $\int_{-\infty}^\infty xf(x)dx = \mathbb{E}[X]$, we get

$$\begin{aligned} 1 &= \int_0^\infty Cxe^{-x}dx \\ &= C\int_0^\infty xe^{-x}dx \\ &= C \\ \implies C &= 1 \end{aligned}$$

(b) Compute the expectation of X.
Solution:
Again, we note that $\int_{-\infty}^\infty x^2f(x)dx = \mathbb{E}[X^2]$. Thus,

$$\begin{aligned} \mathbf{E}[X] &= \int_0^\infty x \cdot xe^{-x}dx \\ &= \int_0^\infty x^2e^{-x}dx \\ &= 2 \end{aligned}$$