ECE 131A
Probability and Statistics
Instructor: Lara Dolecek

Winter 2022 Exam 1
January 24, 2022
Maximum score: 100 points

You have 2 hours to submit your work **directly on Gradescope under the Exam_1 submission link**.
**Please read and carefully follow all the instructions.**

# Instructions

- The exam is accessible from 12:00 am PST on January 24th to 11:59 pm PST on January 24th. Once you open the exam, you will have 2 hours to upload your work (therefore open the exam at least 2 hours before the closing time).
- This exam is open book, open notes. You are allowed to consult your own class notes (homework, discussion, lecture notes, textbook). You are not allowed to consult with each other or solicit external sources for help (e.g., an online forum).
- For each question, start a new sheet of paper. Therefore, the number of pages of your scan should be at least the number of questions. It is ok to write multiple parts of a question on one sheet. Properly erase or cross out any scratch work that is not part of the answer.
- Please submit your exam through the corresponding submission link on Gradescope.
- Make sure to include your **full name** and **UID** in your submitted file.
- Make sure to **show all your work**. Unjustified answers will be at a risk of losing points.
- **Policy on the Academic Integrity**
  "During this exam, you are **disallowed** to contact with a fellow student or with anyone outside the class who can offer a solution e.g., web forum."
  **Please write the following statement on the first page of your answer sheet.**
  You will **lose 10 points** if we can not find this statement.

I $\underline{\quad YourName \quad}$ with UID ____ have read and understood the policy on academic integrity.

0. Please read the Policy on the Academic Integrity on the first page of this exam and follow the instructions.

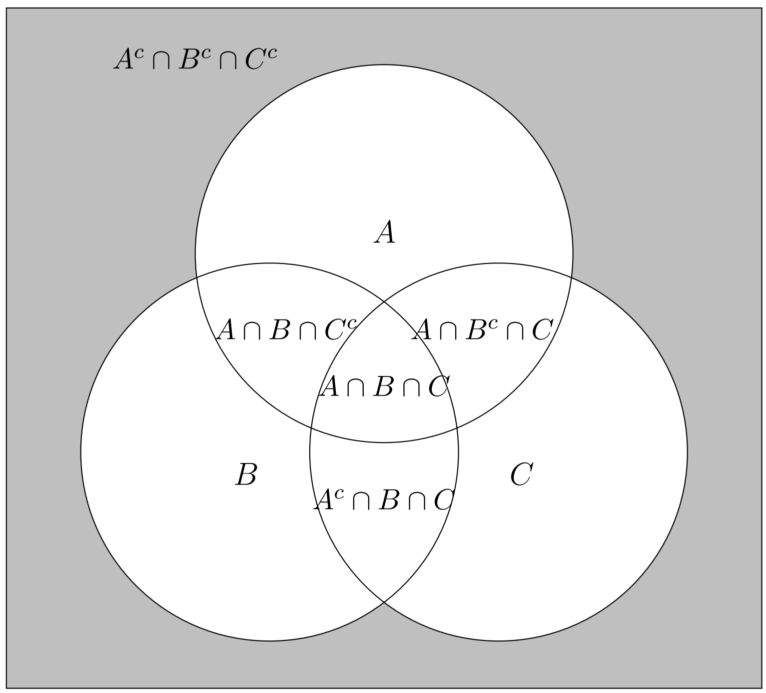
1. (15 pts) Consider three events $A$, $B$, and $C$ defined on the same sample space $S$. Draw a Venn diagram and write the expression for the following events. Use only complements, intersections, and unions operations.

   (a) None of the events occur.

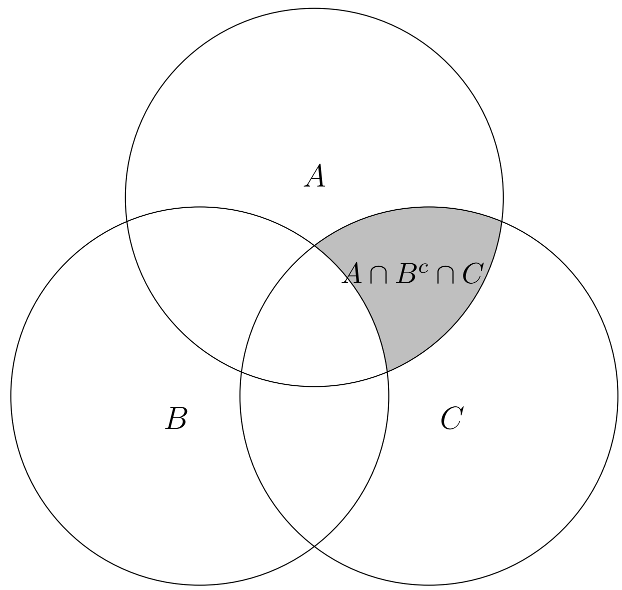
   (b) Events $A$ and $C$ occur, but $B$ does not.

## Solution:

(a) The event we want is $A^c \cap B^c \cap C^c$. The Venn diagram is shown below:




(b) The event we want is $A \cap B^c \cap C$. The Venn diagram is shown below:


$A$
$A \cap B^c \cap C$
$B$
$C$

2. (15 pts) Suppose $P(A) = 1/3$, $P(A \cup B) = 2/3$, $P(A \cap B) = 1/4$. Compute:

   (a) $P(B)$

   (b) $P(A \cap B|B)$

   **Show all your work**.

# Solution:

(a)

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \Rightarrow P(B) &= P(A \cup B) + P(A \cap B) - P(A) \\ &= \frac{2}{3} + \frac{1}{4} - \frac{1}{3} = \frac{7}{12}. \end{aligned}$$

(b)

$$P(A \cap B|B) = \frac{P(A \cap B \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{7}{12}} = \frac{3}{7}$$

3. (15 pts) We draw 8 cards from an ordinary deck of 52 cards. What is the probability that we have exactly three queens?

## Solution:

The total number of outcomes of drawing 8 cards from 52 cards is $\binom{52}{8}$. To have exactly 3 queens we must have drawn 3 cards out of the 4 queens and 5 cards out of the remaining 48 cards. Therefore we have:

$$P[\text{8 drawn cards have exactly 3 queens}] = \frac{\binom{4}{3}\binom{48}{5}}{\binom{52}{8}} = \frac{352}{38675} = 0.009101.$$

4. (15 pts) Mention True or False for each of the following questions.

   **Mentioning the correct answer is worth +3 points, mentioning the incorrect answer is worth −1 points**. Leaving the answer blank is worth 0 points.

   (a) For all events $A, B, C$, we have $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.
   **Solution:** TRUE

   (b) For a random variable $X$, $\text{VAR}(bX) = b^2\text{VAR}(X)$ for all real values of $b$.
   **Solution:** TRUE

   (c) If $P(A \cap B) = 0$, then either $P(A)$ or $P(B)$ must be equal to zero.
   **Solution:** FALSE

   (d) If events $X$ and $Y$ are mutually exclusive, then they are also independent.
   **Solution:** FALSE

   (e) If random variables $X$ and $Y$ are independent and $Var[X] = a$, and $Var[Y] = b$. Then, $Var[X + Y] = a + b$.
   **Solution:** TRUE

5. (20 pts) Suppose $X$ has the following PMF:

$$X = \begin{cases} -5, & \text{with probability } 1/5, \\ 0, & \text{with probability } 3/5, \\ 5, & \text{with probability } 1/5. \end{cases} \tag{1}$$

**Show all your work**.

(a) Compute the mean and variance of $X$.
**Solution:**

$$\mathbf{E}[X] = -5 \times \frac{1}{5} + 0 \times \frac{3}{5} + 5 \times \frac{1}{5} = 0$$

$$\mathbf{Var}(X) = \mathbf{E}[X^2] = 25 \times \frac{1}{5} + 0 \times \frac{3}{5} + 25 \times \frac{1}{5} = 10$$

(b) Compute the mean and variance of $Y = X^2$.
**Solution:**

$$\mathbf{E}[Y] = \mathbf{E}[X^2] = 10$$

$$\mathbf{E}[Y^2] = \mathbf{E}[X^4] = 625 \times \frac{1}{5} + 0 \times \frac{3}{5} + 625 \times \frac{1}{5} = 250$$

Thus

$$\mathbf{Var}(Y) = \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 = 250 - 100 = 150$$

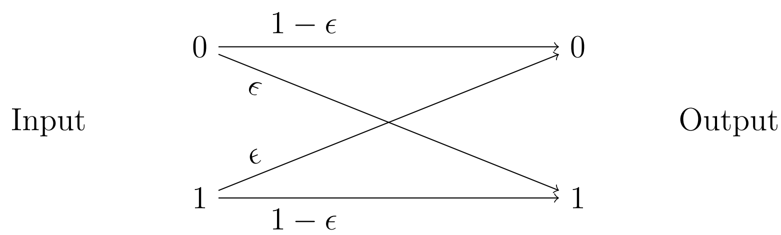
(c) Compute $\mathbf{E}[XY]$.
**Solution:**

$$\mathbf{E}[XY] = \mathbf{E}[X^3] = -125 \times \frac{1}{5} + 0 \times \frac{3}{5} + 125 \times \frac{1}{5} = 0$$

(d) Are $X$ and $Y$ uncorrelated ? Justify your answer.
**Solution:**
Since $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y] = 0$, $X$ and $Y$ are uncorrelated.

(e) Are $X$ and $Y$ independent? Justify your answer.
**Solution:**
X and Y are not independent. To show this, we can provide an example where $P(Y = y|X = x) \neq P(Y = y)$. Clearly, $P(Y = 25|X = 0) = 0$ and $P(Y = 25) = \frac{2}{5}$.

6. (20 pts) A binary symmetric communication channel is shown in the figure. The probability that the output is 1 given that the input is 0 is $\epsilon$. Similarly, probability that the output is 0 given that the input is 1 is $\epsilon$. Assume that input symbols 0 and 1 are chosen for transmission with probabilities $\frac{4}{5}$ and $\frac{1}{5}$, respectively.




(a) Calculate the probability of each output.
**Solution:**

$$\begin{aligned} P(\text{Output } = 0) &= P(\text{Output } = 0|\text{Input } = 0)P(\text{Input } = 0) \\ &+ P(\text{Output } = 0|\text{Input } = 1)P(\text{Input } = 1) \\ &= \frac{4}{5}(1-\epsilon) + \frac{1}{5}\epsilon = \frac{4-3\epsilon}{5} \end{aligned}$$

$$\begin{aligned} P(\text{Output } = 1) &= P(\text{Output } = 1|\text{Input } = 0)P(\text{Input } = 0) \\ &+ P(\text{Output } = 1|\text{Input } = 1)P(\text{Input } = 1) \\ &= \frac{4}{5}\epsilon + \frac{1}{5}(1-\epsilon) + = \frac{1+3\epsilon}{5} \end{aligned}$$

(b) Given that the output was 1, what is the probability that the output was flipped by the channel? **Solution:**
This statement is equivalent to finding $P(\text{Input } = 0|\text{Output } = 1)$. By Bayes rule, we get

$$P(\text{Input } = 0|\text{Output } = 1) = \frac{P(\text{Output } = 1|\text{Input } = 0)P(\text{Input } = 0)}{P(\text{Output } = 1)}.$$

All the necessary terms were calculated in part (a) or are part of the problem statement which gives us the result

$$P(\text{Input } = 0|\text{Output } = 1) = \frac{\frac{4}{5}\epsilon}{\frac{1+3\epsilon}{5}} = \frac{4\epsilon}{1+3\epsilon}$$