I [redacted] w I u I0 [redacted] have read and understood the policy on academic integrity

1) X ~ Gaussian RV $P(-1 < X < 3)$ in term of Q functions

↳ $\mu, \sigma = (0, 1)$ b/c standard

$$P(-1 < X < 3) = P\left(\frac{-1-0}{1} < \frac{X-0}{1} < \frac{3-0}{1} \right]$$

$$= P(-1 < Z < 3)$$

$$= \Phi(3) - \Phi(1)$$

known: $Q(a) = 1 - \Phi a$

$\Phi(a) = 1 - Q(a)$

$$P(-1 < X < 3) = [1 - Q(3)] - [1 - Q(-1)]$$

$$= 1 - Q(3) - 1 + Q(-1)$$

$$\therefore P(-1 < X < 3) = Q(-1) - Q(3)$$

PDF

2) $f_X(x) = \begin{cases} c\, e^{-x}(1+x) & x \geq 0 \\ 0 & \text{else} \end{cases}$

2a) constant c

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\cancel{\int_{-\infty}^{0} 0\,dx} + \int_{0}^{\infty} c\, e^{-x}(1+x)\,dx = 1$$

$$\int_{0}^{\infty} c\, e^{-x}(1+x)\,dx = 1$$

$$c\int_{0}^{\infty} e^{-x}(1+x)\,dx = 1$$

$u = x+1 \qquad v = -e^{-x}$

$du = 1 \qquad dv = e^{-x}\,dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^\infty e^{-x}(1+x)\,dx = (x+1)(-)(e^{-x})\Big|_0^\infty - \int_0^\infty -e^{-x}(1\,dx)$$

$$= (x+1)(-)(e^{-x})\Big|_0^\infty + \int_0^\infty e^{-x}\,dx$$

$$= (x+1)(-e^{-x})\Big|_0^\infty + (-e^{-x})\Big|_0^\infty$$

$$= -e^{-x}x - e^{-x}\Big|_0^\infty - e^{-x}\Big|_0^\infty$$

$$= -e^{-x}x - 2e^{-x}\Big|_0^\infty$$

$$= -e^{-x}x - 2e^{-x} \Big|_0^{\infty}$$

$$= \cancel{\frac{\infty}{-e^{\infty}}}^{0} - \cancel{\frac{2}{e^{\infty}}}^{0} + \cancel{\frac{0}{e^{0}}}^{0} + \frac{2}{e^{0}}\ 1$$

$\int_0^\infty e^{-x}(1+x)\,dx = 2$

Recall $c \int_0^\infty e^{-x}(1+x)\,dx = 1$

$$c\,(2) = 1$$

$$c = \frac{1}{2}$$

2b) $\mathbb{E}[X]$

continuous X: $\mathbb{E}[X] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx$

$$f_X(x) = \begin{cases} c\,e^{-x}(1+x) & x \geq 0 \\ 0 & \text{else} \end{cases}$$

$$g(x) = x$$

$$\mathbb{E}[X] = \int_{-\infty}^{0} x\,(0)\,dx + \int_0^\infty x\left(\tfrac{1}{2}\right)e^{-x}(1+x)\,dx$$

$$= \int_0^\infty x\left(\tfrac{1}{2}\right)e^{-x}(1+x)\,dx$$

$$\mathbb{E}[X] = \frac{1}{2}\int_0^\infty x\,e^{-x}(1+x)\,dx$$

Aside:

$$\int_0^\infty x e^{-x} (1+x)\, dx$$

$$\int u\, dv = uv - \int v\, du$$

$u = x(x+1)$ $\quad$ $v = -e^{-x}$

$= x^2 + x$

$du = 2x+1\, dx$ $\quad$ $dv = e^{-x}\, dx$

$$= (x^2+x)(-e^{-x}) \Big|_0^\infty - \int_0^\infty (-e^{-x})(2x+1)\, dx$$

$u = 2x+1$ $\quad$ $v = e^{-x}$

$du = 2\, dx$ $\quad$ $dv = -e^{-x}\, dx$

$$= (x^2+x)(-e^{-x}) \Big|_0^\infty - \left[(2x+1)(e^{-x}) \Big|_0^\infty - \int_0^\infty e^{-x}\, 2\, dx \right]$$

$$= (x^2+x)(-e^{-x}) \Big|_0^\infty - \left[(2x+1)(e^{-x}) \Big|_0^\infty + 2e^{-x} \Big|_0^\infty \right]$$

$$= (x^2+x)(-e^{-x}) - (2x+1)e^{-x} - 2e^{-x} \Big|_0^\infty$$

$$= -\frac{x^2}{e^x} - \frac{x}{e^{-x}} - \frac{2x+1}{e^x} - \frac{2}{e^x} \Big|_0^\infty$$

$$= -\cancel{\frac{\infty^2}{e^\infty}} - \cancel{\frac{\infty}{e^\infty}} - \cancel{\frac{2(\infty)+1}{e^\infty}} - \cancel{\frac{2}{e^\infty}}$$

$$+ \frac{0}{e^0} + \frac{0}{e^0} + \frac{0+1}{e^0} + \frac{2}{e^0}$$

$= 3$

recall

$$\mathbb{E}[X] = \frac{1}{2} \underbrace{\int_0^\infty x e^{-x} (1+x)\, dx}_{\hookrightarrow\ 3}$$

$$\mathbb{E}[X] = \frac{1}{2}(3)$$

$$\therefore\ \mathbb{E}[X] = \frac{3}{2}$$

3) Geometric RV X parameter p

Find $\Phi(X)$ and $\mathbb{E}[X^2]$

$$\Phi_X(\omega) = \mathbb{E}\left[e^{j\omega X}\right]$$ Note: $g(x) = e^{j\omega x}$

Geometric RV(p)

is the number of trials until the first success of independent Bernoli (p) trials

$$\boxed{\mathbb{E}[X] = \frac{1}{p} \qquad \mathrm{VAR}(X) = \frac{1-p}{p^2}}$$

p

Discrete X : $\mathbb{E}[X] = \sum_i g(x_i) \cdot P(X = x_i)$

$$g(x_i) = p(1-p)^k$$ let $1-p = q$

$$= p\,q^k$$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} p\,q^k\, e^{j\omega k}$$

$$= p \sum_{k=0}^{\infty} (1-p)^k e^{j\omega k}$$

$$= p \sum_{k=0}^{\infty} \left((1-p) e^{j\omega} \right)^k$$

$$\sum_{k=0}^{\infty} \left[e^{j\omega} (1-p) \right]^k = \frac{1}{1-(1-p)e^{j\omega}}$$

$$= \frac{p}{1-(1-p)e^{j\omega}}$$

$$\Phi_X(\omega) = \frac{p}{1-qe^{j\omega}}$$

$$\therefore \Phi_X(\omega) = \frac{p}{1-qe^{j\omega}}$$

$$= \frac{p}{1-(1-p)e^{j\omega}}$$

$$\frac{d\Phi_X(\omega)}{d\omega} = \frac{d}{d\omega} \frac{p}{1-qe^{j\omega}}$$

$$= p \frac{d}{d\omega} (1-qe^{j\omega})^{-1}$$

$$= p(-1)(1-qe^{j\omega})^{-2}(0-qe^{j\omega}(j))$$

$$= p\cancel{(-1)}(1-qe^{j\omega})^{-2}(\cancel{-}qje^{j\omega})$$

$$= \frac{jpqe^{j\omega}}{(1-qe^{j\omega})^2}$$

$$\frac{d^2\Phi_X(\omega)}{d\omega^2} = \frac{d}{d\omega}\left(\frac{jpqe^{j\omega}}{(1-qe^{j\omega})^2}\right)$$

$$\frac{d^2 \Phi_x(\omega)}{d\omega^2} = \frac{d}{d\omega}\left(\frac{jpqe^{j\omega}}{(1-qe^{j\omega})^2} \right)$$

$$= \frac{(-1)(1-qe^{j\omega})^2 (jpq\, e^{j\omega})(j) + (-1)(jpqe^{j\omega})\, 2(1-qe^{j\omega})^1 (+qe^{j\omega} j)}{\left((1-qe^{j\omega})^2 \right)^2}$$

$$= \frac{(-1)(1-qe^{j\omega})^2(pq\, e^{j\omega}) + (-1)(pq\, e^{j\omega})(2)(1-qe^{j\omega})(q\, e^{j\omega})}{(1-qe^{j\omega})^4}$$

$$\mathbb{E}[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega) \Big|_{\omega=0}$$

$$\mathbb{E}[X^2] = \frac{1}{j^2} \frac{d^2}{d\omega^2} \Phi_x(\omega) \Big|_{\omega=0}$$

Note: $\frac{d^2}{d\omega^2} \Phi_x(\omega) \Big|_0$

$$= \frac{(-1)(1-q\cancel{e^{j\omega}})^2(pq\, \cancel{e^{j\omega}}) + (-1)(pq\, \cancel{e^{j\omega}})(2)(1-q\cancel{e^{j\omega}})(q\, \cancel{e^{j\omega}})}{(1-q\cancel{e^{j\omega}})^4}$$

$$= \frac{(-1)(1-q)^2(pq) + (-1)(pq)(2)(1-q)(q)}{(1-q)^4}$$

$$= (-1)\left[\frac{(1-q)^2(pq) + (pq)(2)(1-q)(q)}{(1-q)^4} \right)$$

$$\mathbb{E}[X^2] = \cancel{(-1)} \quad \cancel{(-1)} \left[\frac{(1-q)^2(pq) + (pq)(2)(1-q)(q)}{(1-q)^4} \right)$$

$$= \frac{(1-q)^2(pq) + pq(2)(1-q)(q)}{(1-q)^4}$$

$$= \frac{pq(1-q)\left[(1-q) + 2q\right]}{(1-q)^4}$$

$$\mathbb{E}[X^2] = \frac{pq(1+q)}{(1-q)^3}$$

4) $X \sim \mathcal{N}(0, 4)$ $\quad Y = \begin{cases} X & X \geq 0 \\ X^2 & X < 0 \end{cases}$

Gaussian RV $X \sim \mathcal{N}(\mu, \sigma^2)$

x has gaussian distribution w/ the parameters μ & σ^2

$$f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

$$Y = \begin{cases} X & x \geq 0 \\ X^2 & x < 0 \end{cases}$$

$$F_Y(y) = \begin{cases} F_X(y) \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(y) & y \geq 0 \\ \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) & y < 0 \end{cases}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-(x-\mu)^2/2\sigma^2} \qquad f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x^2)/8}$$

$$f_Y(y) = \begin{cases} \dfrac{e^{-y^2/8}}{2\sqrt{2\pi}} & y \geq 0 \\ \dfrac{e^{-y/8}}{4\sqrt{2\pi y}} + \dfrac{e^{y/8}}{4\sqrt{2\pi y}} & y < 0 \end{cases}$$

$$\therefore \quad f_Y(y) = \begin{cases} \dfrac{e^{-y^2/8}}{2\sqrt{2\pi}} & y \geq 0 \\[2ex] \dfrac{\left(e^{-y/8} + e^{y/8}\right)}{\sqrt{2\pi y}} & y < 0 \end{cases}$$

5) X - uniform rv $[-a, a]$ $a > 0$

5a) $\dfrac{1}{b-a} = \dfrac{1}{a-(-a)} = \dfrac{1}{2a}$

$$f_X(x) = \begin{cases} \dfrac{1}{2a} & -a \le x \le a \\ 0 & \text{elsw} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x+a}{2a} & -a \le x < a \\ 1 & x \ge a \end{cases}$$

$$P(X > c) = 1 - P(X \le c)$$

$$= 1 - F_X(c)$$

$\downarrow$

$$1 - F_X(c) = \begin{cases} 1 - 0 & c < a \\ 1 - \dfrac{c+a}{2a} & -a \le c < a \\ 1 - 1 & c \ge a \end{cases}$$

$$1 - F_X(c) = \begin{cases} 1 & c < a \\ \frac{a-c}{2a} & -a \leq c < a \\ 0 & c \geq a \end{cases}$$

Note: $1 - \frac{c+a}{2a} = \frac{2a}{2a} - \frac{(c+a)}{2a} = \frac{a-c}{2a}$

$$\therefore P(X > c) = \begin{cases} 1 & c < a \\ \frac{a-c}{2a} & -a \leq c < a \\ 0 & c \geq a \end{cases}$$

5b) Bound $P(X > c)$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\mathbb{E}[X] = \frac{a-a}{2} = 0$$

$$VAR(X) = \frac{(b-a)^2}{12}$$

$$VAR(X) = \frac{(a-(-a))^2}{12} = \frac{4a^2}{12} = \frac{a^2}{3}$$

$$p(x > c) = p(x \geq c) - p(x > c)$$

$$p(x > c) \leq p(x \geq c) \leq \frac{\sigma^2}{a'^2}$$

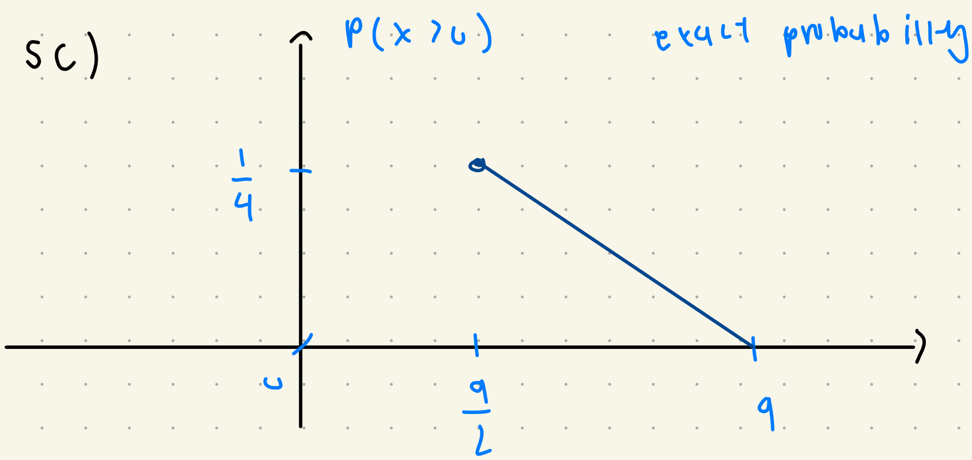
Note $a' = c$

$$\sigma^2 = VAR = \frac{a^2}{3}$$

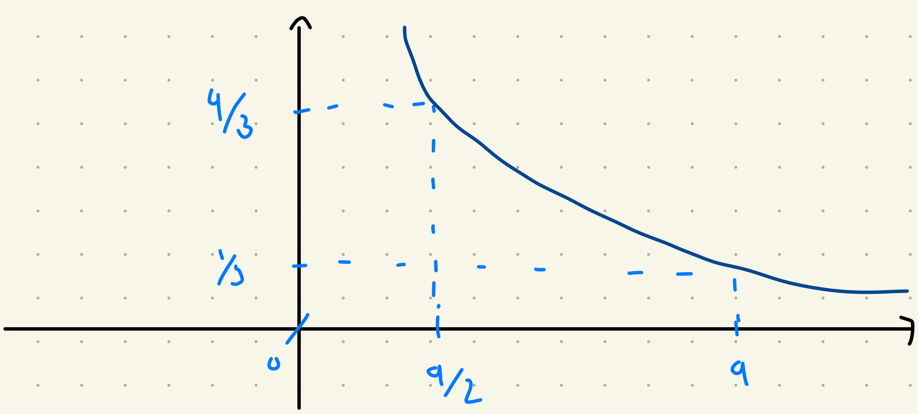
$$\leq \frac{a^2/3}{c^2}$$

$$\leq \frac{a^2}{3c^2}$$

$$\therefore \; p(x > c) \leq \frac{a^2}{3c^2}$$



chebyshev upper bound graph



6) $X \sim \exp(\lambda)$

$Y = X + 1$

$X \sim \exp(\lambda)$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$\mathrm{VAR}(X) = \frac{1}{\lambda^2}$

6a) $\mathbb{E}[Y] = \mathbb{E}[X+1] = \mathbb{E}[X] + \mathbb{E}[1]$

$= \mathbb{E}[X] + 1$

$= \frac{1}{\lambda} + 1 = \frac{\lambda+1}{\lambda}$

$\mathbb{E}[Y] = \frac{\lambda+1}{\lambda}$

$\mathrm{VAR}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$= \mathbb{E}[(X+1)^2] - \left(\frac{\lambda+1}{\lambda}\right)^2$

Aside $\mathbb{E}((X+1)^2) = \mathbb{E}[X^2 + 2X + 1]$

$= \mathbb{E}[X^2] + \mathbb{E}[2X] + \mathbb{E}[1]$

$= \mathbb{E}[X^2] + 2\mathbb{E}[X] + 1$

$= \frac{2}{\lambda^2} + 2\frac{1}{\lambda} + 1$

$$= \mathbb{E}[(X+1)^2] - \left(\frac{\lambda+1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} + \frac{2}{\lambda} + 1 - \frac{(\lambda+1)^2}{\lambda^2}$$

$$= \frac{2}{\lambda^2} + \frac{2\lambda}{\lambda^2} + \frac{\lambda^2}{\lambda^2} - \frac{\lambda^2 + 2\lambda + 1}{\lambda^2}$$

$$= \frac{2-1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

$$\mathrm{VAR}(Y) = \frac{1}{\lambda^2}$$