Midterm Solutions Version A Ahmed Hareedy

1. (10 pts) Consider the events A_1, A_2, and A_3. Recall that:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

Use this property to prove that:

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3).$$

Sol. $P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup (A_2 \cup A_3))$

$= P(A_1) + P(A_2 \cup A_3) - P(A_1 \cap (A_2 \cup A_3))$

$= P(A_1) + [P(A_2) + P(A_3) - P(A_2 \cap A_3)]$

$\quad - P((A_1 \cap A_2) \cup (A_1 \cap A_3))$

$= P(A_1) + P(A_2) + P(A_3) - P(A_2 \cap A_3)$

$\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P((A_1 \cap A_2) \cap (A_1 \cap A_3))$

$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3)$

$\quad - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$. ■

(The same for Version B)

2. (12 pts) Suppose Y is a Laplacian random variable, which is defined by the PDF:

$$f_Y(y) = he^{-4\alpha|y|}, \quad -\infty < y < \infty, \ \alpha > 0.$$

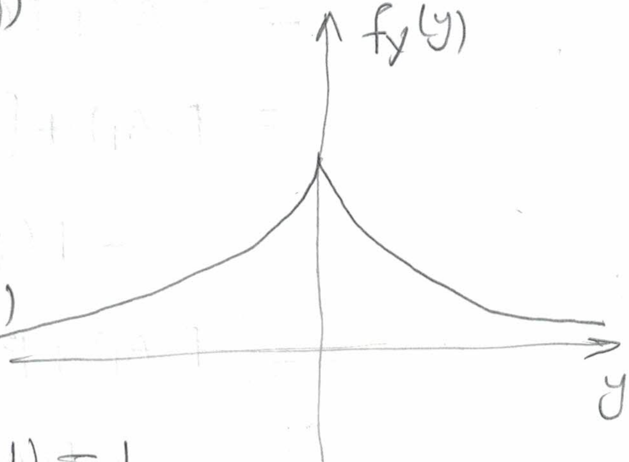
(a) (4 pts) Determine the value of h as a function of α.

(b) (8 pts) Determine the CDF of the random variable Y.

Sol.

$$f_Y(y) = he^{-4\alpha|y|} = \begin{cases} he^{4\alpha y}, & -\infty < y < 0 \\ he^{-4\alpha y}, & 0 \leq y < \infty \end{cases}$$

(a) $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$ (PDF property)



$$\int_{-\infty}^{0} he^{4\alpha y}\,dy + \int_{0}^{\infty} he^{-4\alpha y}\,dy = 1$$

$$\Rightarrow 2\int_{0}^{\infty} he^{-4\alpha y}\,dy = 1 \quad \text{(by symmetry)}$$

$$\Rightarrow 2h \left.\frac{e^{-4\alpha y}}{4\alpha}\right|_{\infty}^{0} = 1 \quad \Rightarrow \frac{h}{2\alpha}(1) = 1$$

$$\Rightarrow \boxed{h = 2\alpha} \quad (\text{For Version B} \rightarrow \boxed{h = 3\alpha})$$

(b) Region 1 $\quad -\infty < y < 0$

$$F_Y(y) = \int_{-\infty}^{y} 2\alpha e^{4\alpha y}\,dy = 2\alpha \left.\frac{e^{4\alpha y}}{4\alpha}\right|_{-\infty}^{y} = \frac{1}{2}e^{4\alpha y}$$

Region 2 $\quad 0 \leq y < \infty$

$$F_Y(y) = \int_{-\infty}^{0} 2\alpha e^{-4\alpha y}\,dy + \int_{0}^{\infty} 2\alpha e^{-4\alpha y}\,dy = \frac{1}{2} + 2\alpha \left.\frac{e^{-4\alpha y}}{4\alpha}\right|_{\infty}^{0}$$

$$= \frac{1}{2} + \frac{1}{2}(1 - e^{-4\alpha y}) = 1 - \frac{1}{2}e^{-4\alpha y}$$

(For Version B, use 3α instead of 2α)

3. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) Suppose X is a discrete random variable. The expected value of X, $E(X)$, has to be a value X can take.

TRUE FALSE

(b) Consider the events A, B, and C. Then, $P(A, B \mid C) = P(A \mid B, C)P(B)$.

TRUE FALSE

(c) Suppose X and Y are dependent random variables. Then, $E(X + Y) = E(X) + E(Y)$.

TRUE FALSE

(d) Consider a random variable X that is not a constant. It is possible for that random variable to have zero mean, i.e., $E(X)$ can be 0, but it is not possible for it to have zero variance, i.e., $VAR(X) \neq 0$.

TRUE FALSE

(e) For $x > 0$, $Q(x) + Q(-3x/2) < 1$, where Q refers to the Q-function.

TRUE FALSE

(For Version B → F F F T T)

4. (10 pts) Suppose X is a Gaussian random variable with mean $m_X = 3$, and variance $\sigma_X^2 = 4$. Let Z be another random variable that is defined as $Z = 5X + 1$. Express the following probabilities in terms of the Q-function:

(a) (4 pts) $P(X < 2)$.

(b) (6 pts) $P(-4 \le Z \le 16)$.

Sol. $X \sim \mathcal{N}(3,4)$

(a) $P(X<2) = P\left(\frac{X-3}{2} < \frac{2-3}{2}\right)$

$= P\left(X_s < \frac{-1}{2}\right) = F_{X_s}\left(\frac{-1}{2}\right) = 1-Q\left(\frac{-1}{2}\right) = Q\left(\frac{1}{2}\right)$

(b) $P(-4 \leq Z \leq 16) = P(-4 \leq 5X+1 \leq 16)$

$= P\left(\frac{-5}{5} \leq X \leq \frac{15}{5}\right) = P(-1 \leq X \leq 3)$

$= P\left(\frac{-1-3}{2} \leq X_s \leq \frac{3-3}{2}\right) = P(-2 \leq X_s \leq 0)$

$= F_{X_s}(0) - F_{X_s}(-2) = Q(-2) - Q(0)$

$= Q(0) - Q(2) = \frac{1}{2} - Q(2)$

For Version B $\rightarrow$ (a) $1-Q\left(\frac{1}{2}\right)$

(b) $Q\left(\frac{1}{2}\right) - Q\left(\frac{5}{2}\right)$

5. (12 pts) You are a competitor on a game show, and you are given a choice of n doors. Behind one door there is a car; behind the other $n-1$ doors are goats. You pick a door, and the game show host, who knows what is behind each of the doors, opens another door, which has a goat behind it. The host asks you, "Do you want to change your door?". Assuming that you want the car:

(a) (3 pts) Find the probability that you win the car if you stick to your door choice.

(b) (7 pts) Find the probability that you win the car if you change your door choice.

(c) (2 pts) Comment on your results when n is large.

Without loss of generality, assume the car is behind Door 1!

Sol. We build the following table: Conditioned

	You select	Host selects	P(stay & win)	P(change & win)
(1/n)	1	2, 3, ..., n	1	0
(1/n)	2	3, 4, ..., n	0	1/(n−2)
(1/n)	3	2, 4, ..., n	0	1/(n−2)
⋮	⋮	⋮	⋮	⋮
(1/n)	n	2, 3, ..., n−1	0	1/(n−2)

(a) $P(\text{stay \& win}) = \left(\frac{1}{n}\right)(1) + (0)(n-1) = \boxed{\frac{1}{n}}$

(b) $P(\text{change \& win}) = (0)(1) + \frac{1}{n(n-2)}(n-1) = \boxed{\frac{n-1}{n(n-2)}}$

(c) For large $n \rightarrow P(\text{stay \& win}) \approx P(\text{change \& win}) \approx \frac{1}{n}$

The advantage of changing vanishes.

(The same for Version B)

6. (17 pts) There are n boxes labeled 1, 2, ..., n. Furthermore, there are n balls labeled 1, 2, ..., n. Consider the case of $n \geq 2$. You place the balls into the boxes. Each ball has an equal probability of being placed into any box. Find the probability that after placing all the balls:

(a) (5 pts) Box number 1 is (and possibly others are) empty.

(b) (12 pts) Exactly one box is empty.

Sol. This is a combinatorics problem..

(a) Denominator and numerator :

Number of ways to select a box out of n, n times $= n^n$

Number of ways to select a box out of $(n-1)$, n times

$= (n-1)^n \rightarrow \boxed{P = \frac{(n-1)^n}{n^n}}$

(b) The denominator stays the same $\rightarrow n^n$

Pick 1 box out of n to be empty $\rightarrow \binom{n}{1} = n$

Pick 1 box out of $(n-1)$ to have 2 balls $\rightarrow \binom{n-1}{1} = n-1$

Pick 2 balls for that box (out of n) $\rightarrow \binom{n}{2}$

Distribute the remaining $(n-2)$ balls $\rightarrow (n-2)!$

$$\Rightarrow P = \frac{(n)(n-1)\binom{n}{2}(n-2)!}{n^n} = \boxed{\frac{\binom{n}{2}\, n!}{n^n}}$$

(The same for Version B)

7. (10 pts) Carl and Martin are professional basketball players. They are training on the mid-court shot. Each one of them keeps trying to make the mid-court shot until he succeeds. The probability that Carl takes 3 trials to make the mid-court shot is 81/1000, while the probability that Martin takes 4 trials to make the mid-court shot is 1024/10000. Which player of the two is a better mid-court shooter? Justify your answer with probabilistic arguments.

Sol. Carl: $P_c(1-P_c)^2 = \frac{81}{1000}$

$\therefore P_c(1-P_c)^2 = \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^2 \rightarrow P_c = \frac{1}{10}$

Martin: $P_m(1-P_m)^3 = \frac{1024}{10000}$

$\therefore P_m(1-P_m)^3 = \left(\frac{2}{10}\right)\left(\frac{8}{10}\right)^3 \rightarrow P_m = \frac{2}{10}$

Since $P_m > P_c$

$\therefore$ Martin is the better mid-court shooter.

(The same for Version B)

8. (14 pts + 5 bonus) You are playing a game in which you are tossing a fair coin n times. You select n initially. Each coin toss, irrespective of the coin toss outcome, costs you 7 points. The prize for getting X heads out of all your tosses is given by $4X^2 - 4X + 12$ points. Your number of net points is counted after you finish your n tosses. You win the game if your number of net points is strictly more than zero.

   (a) (9 pts) Find your expected number of net points as a function of n.

   (b) (5 pts) What is the range of n (number of tosses) you need to avoid in order to expect to win the game?

   (c) (5 pts) **Bonus:** What is the least expected number of net points you can get in this game?

Sol. Net points $N = 4X^2 - 4X + 12 - 7n$, X is binomial

(a) $E(N) = 4E(X^2) - 4E(X) + 12 - 7n$

$E(X) = np = \frac{n}{2}$, $E(X^2) = V(X) + (E(X))^2 = \frac{n}{4} + \frac{n^2}{4}$

$\Rightarrow E(N) = 4\left[\frac{n}{4} + \frac{n^2}{4}\right] - 4\left(\frac{n}{2}\right) + 12 - 7n$

$= n^2 + n - 2n - 7n + 12 = \boxed{n^2 - 8n + 12}$

(b) $E(N) = n^2 - 8n + 12 = (n-2)(n-6)$

For n between 2 & 6, $E(N)$ is negative

$\Rightarrow$ Avoid $\boxed{2 \leq n \leq 6}$ ($E(N) = 0$ should be avoided)

(c) $\frac{dE(N)}{dn} = 2n - 8 = 0 \Rightarrow n = 4$ for $E(N)|_{min}$

$\Rightarrow E(N)|_{min} = (4)^2 - 8(4) + 12 = 16 - 32 + 12 = \boxed{-4}$

(The same for Version B)