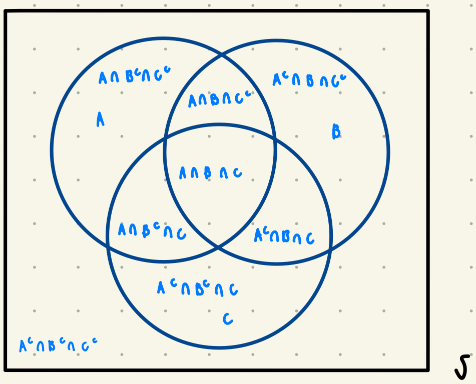
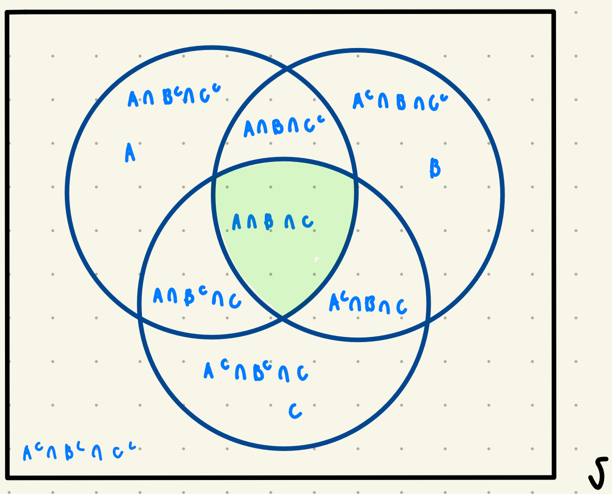
I, [redacted], with UID: [redacted] have read and understood the policy on academic integrity

1)



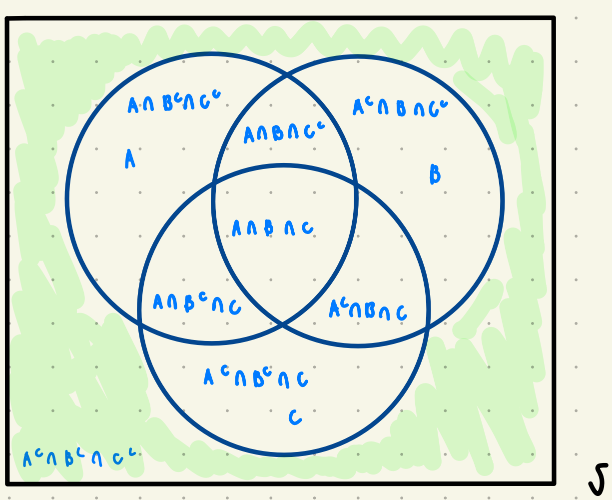

1a) all three events occur




Expression

$A \cap B \cap C$

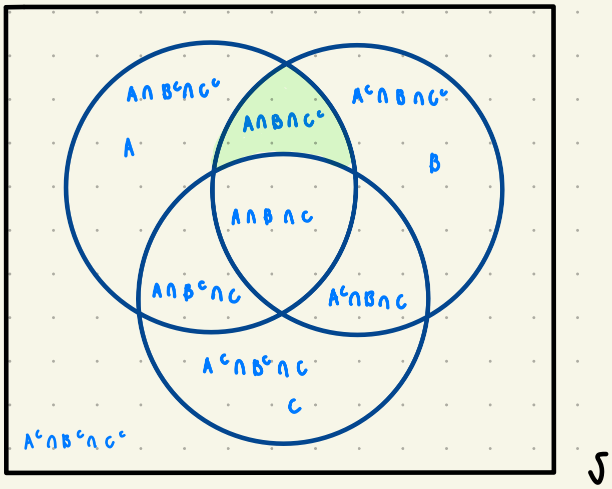
1b) None of the events occur




Expression

$A^c \cap B^c \cap C^c$

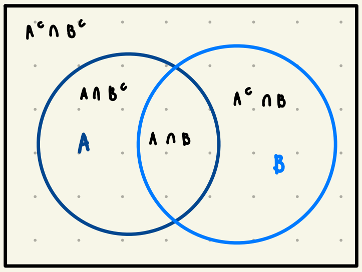
1c) A and B occur but C does not




Expression

$A \cap B \cap C^c$

2) $P(A) = \frac{2}{10}$ $P(B) = \frac{3}{10}$ $P(A \cup B) = \frac{4}{10}$




2a) $P(A \cap B)$

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\text{solve for}}$$

$$P(A \cap B) = \underbrace{P(A) + P(B) - P(A \cup B)}_{\text{all values given}}$$

$$= \frac{2}{10} + \frac{3}{10} - \frac{4}{10}$$

$$= \frac{1}{10}$$

$$P(A \cap B) = \frac{1}{10}$$

2b) P(A|B)

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \begin{array}{l} \leftarrow \text{from part 2a} \\ \leftarrow \text{given} \end{array}$$

$$= \frac{1/10}{3/10}$$

$$= \frac{1}{3}$$

$$P(A \mid B) = \frac{1}{3}$$

2c) P(B|A)

Bayes Rule

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \mid B) \cdot P(B)}{P(A)}$$

$$= \frac{1/\cancel{3} \cdot \cancel{3}/\cancel{10}}{2/\cancel{10}}$$

$$= \frac{1}{2}$$

$$P(B \mid A) = \frac{1}{2}$$

3) PMF of X

$$X = \begin{cases} -2 & \text{w.p. } 1/4 \\ 0 & \text{w.p. } 1/2 \\ 2 & \text{w.p. } 1/4 \end{cases}$$

3a) mean and variance of X

$$\mathbb{E}[g(X)] = \sum_{x_i \in S_X} g(x_i) \cdot p_X(x_i)$$

$$\mathbb{E}[X] = m_X \qquad \text{mean}$$

$\downarrow$

$$m_X = -2\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right)$$

$$= -\cancel{\frac{1}{2}} + 0 + \cancel{\frac{1}{2}}$$

$$= 0$$

mean $= m_X = 0$

$$\mathbb{E}[X^2] - m_X^2 = \text{VAR}(X)$$

$\downarrow$

$m_X =$ mean $= 0$

$m_X^2 = 0$

$\downarrow$

Find $\mathbb{E}[X^2]$

$$\mathbb{E}[X^2] = (-2)^2\left(\frac{1}{4}\right) + 0^2 \frac{1}{2} + 2^2\left(\frac{1}{4}\right)$$

$$= 4\left(\frac{1}{4}\right) + 0 + 4\left(\frac{1}{4}\right)$$

$$= 1 + 0 + 1$$

$$= 2$$

$$\text{VAR}[X] = \mathbb{E}[X^2] - m_X^2$$

$$= 2 - 0$$

$$= 2$$

$\text{VAR}[X] = 2$

36) $Y = X^2$ mean and variance of $Y$

Mean $m_y = \mathbb{E}[Y]$

$m_y = \mathbb{E}[Y] = \mathbb{E}[X^2]$

found in part 34

$= 2$

$m_y = 2$

variance

$VAR(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

Note: $\mathbb{E}[Y] = 2$

$(\mathbb{E}[Y])^2 = 4$

Find $\mathbb{E}[Y^2]$

$$\mathbb{E}[Y^2] = \mathbb{E}[(X^2)^2]$$

$$= \mathbb{E}[X^4]$$

$$= (-2)^4\left(\frac{1}{4}\right) + 0^4\left(\frac{1}{2}\right) + 2^4\left(\frac{1}{4}\right)$$

$$= 16\left(\frac{1}{4}\right) + 0 + 16\ \frac{1}{4}$$

$$= 4 + 0 + 4$$

$$= 8$$

$VAR(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$VAR(Y) = 8 - 4 = 4$

$VAR(Y) = 4$

3c) compute $\mathbb{E}[XY]$ Note $Y = X^2$

$$\mathbb{E}[XY] = \mathbb{E}[X \cdot X^2]$$
$$= \mathbb{E}[X^3]$$
$$= (-2)^3 \frac{1}{4} + 0^3 \frac{1}{2} + 2^3 \left(\frac{1}{4}\right)$$
$$= -8\left(\frac{1}{4}\right) + 0 + 8\left(\frac{1}{4}\right)$$
$$= \cancel{-2} + 0 + \cancel{2}$$
$$= 0$$

$\mathbb{E}[XY] = 0$

3d) Are X and Y uncorrelated?

uncorrelated
$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

part 3a $\mathbb{E}[X] = 0$

part 3b $\mathbb{E}[Y] = 0$

part 3c $\mathbb{E}[XY] = 0$

$$0 = 0 \cdot 0$$
$0 = 0$ ✓ statement is valid

$\therefore$ X and Y are uncorrelated

4) 10 balls 3 red balls
7 blue balls

Draw 3 one at a time w/o replacement
What is probability getting all 3 blue balls

w/o replacement w/o ordering

Let A be the event 3 blue balls are choosen

$$P(A) = \frac{\binom{7}{3}\binom{3}{0}}{\binom{10}{3}}$$

choosing all 3 blue and 0 red

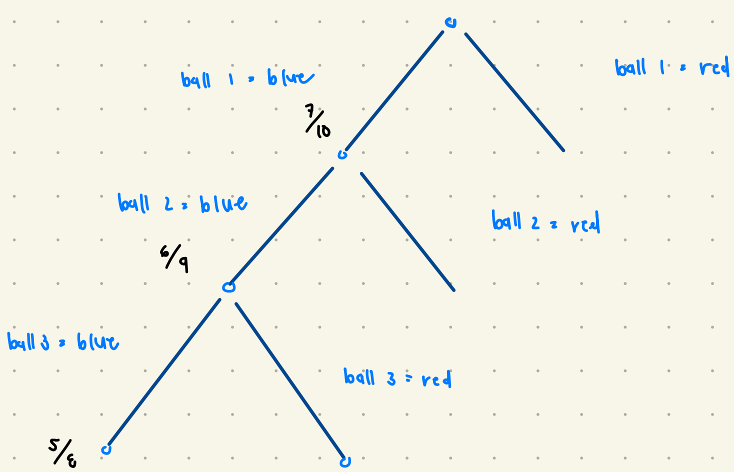
all possible combos

$$= \frac{\frac{7!}{(7-3)!\,3!} \cdot \frac{3!}{(3-0)!\,0!}}{\frac{10!}{(10-3)!\,3!}} = \frac{\frac{\cancel{7!}^{\,7\cdot6\cdot5}}{\cancel{4!}\,3!} \cdot \cancel{\frac{3!}{3!\,0!}}^{\,1}}{\frac{\cancel{10!}^{\,10\cdot9\cdot8}}{\cancel{7!}\,3!}} = \frac{\frac{7\cdot\cancel{6}\cdot5}{\cancel{3}\cdot\cancel{2}\cdot1}\cdot 1}{\frac{\cancel{10}\cdot\cancel{9}^{\,3}\cdot8}{\cancel{3}\cdot\cancel{2}\cdot1}}$$

$$P(A) = \frac{7}{3\cdot8} = \frac{7}{24}$$

$$\boxed{P(A) = \frac{7}{24}}$$

sanity check




all blue balls

$$P(A) = \frac{7}{\cancel{10}_{2}} \cdot \frac{\cancel{6}}{\cancel{9}_{3}} \cdot \frac{\cancel{5}}{\cancel{8}_{4}} = \frac{7}{24}$$

5) biased coin w/ heads being $1/4$
throw coin 3 times

5a) sample space of random experiment

total outcomes $= 2^n$ w/ $n =$ # of tosses
$|S| = 2^3 = 8$

sample space

$$\boxed{S = \left\{ \begin{array}{l} (H\ H\ H),\ (H\ H\ T),\ (H\ T\ H),\ (H\ T\ T) \\ (T\ T\ T),\ (T\ T\ H),\ (T\ H\ T),\ (T\ H\ H) \end{array} \right\}}$$

w/ $|S| = 8$

b) probability of exactly 1 head

let X be the event of exactly 1 head in 3 tosses

$$X = \{ \underset{X_1}{(HTT)},\ \underset{X_2}{(THT)},\ \underset{X_3}{(TTH)} \}$$

$$P(X) = P(X_1) + P(X_2) + P(X_3)$$

Note: H = getting heads $\quad P(H) = 1/4$
T = getting tails $\quad P(T) = 3/4$

$$P(X_1) = (HTT) = \tfrac{1}{4} \cdot \tfrac{3}{4} \cdot \tfrac{3}{4} = \frac{9}{64}$$

$$P(X_2) = (THT) = \tfrac{3}{4} \cdot \tfrac{1}{4} \cdot \tfrac{3}{4} = \frac{9}{64}$$

$$P(X_3) = (TTH) = \tfrac{3}{4} \cdot \tfrac{3}{4} \cdot \tfrac{1}{4} = \frac{9}{64}$$

$$P(X) = \frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64}$$

$$\boxed{P(X) = \frac{27}{64}}$$

6) probability of getting a toy from cereal box is $1/4$

6a) expected # of cereal boxes to open until getting 1st toy

Geometric RV
↓
b/c # of boxes until we get a toy (toy = success)

Let X be the event of getting first toy

Note: $p = 1/4$

Expectation of X

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot p(1-p)^{k-1}$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

Recall $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $\qquad \sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$

$$\mathbb{E}[X] = \frac{1}{p} = \frac{1}{1/4} = 4$$

↓

Expected to open 4 boxes to find 1st toy

6b) expected # of cereal boxes to open until getting 2nd toy

Let X be the event of getting a 2nd toy

$$\mathbb{E}[X] = \sum_{k=2}^{\infty} k \cdot P(X=k)$$

$$= \sum_{k=2}^{\infty} k(k-1)\, p^2 (1-p)^{k-2}$$

$$= p^2 \sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2}$$

↓

Differentiation trick

↓

after differentiation trick

$$= p^2 \sum_{k=2}^{\infty} (1-p)^k \quad \frac{d^2}{dp^2}$$

$$= p^2 \frac{d^2}{dp^2} \sum_{k=2}^{\infty} (1-p)^k$$

Note: Geometric Sum Formula

$$\sum_{k=2}^{\infty} (1-p)^k = \frac{(1-p)^2}{1-(1-p)}$$

$$= p^2 \frac{d^2}{dp^2} \frac{(1-p)^2}{\cancel{1} - (\cancel{1}-p)}$$

$$= p^2 \frac{d^2}{dp^2} \frac{1-2p+p^2}{p}$$

$$= p^2 \frac{d^2}{dp^2} \frac{1}{p} - 2 + p$$

$$= p^2 \frac{d}{dp} -\frac{1}{p^2} - 0 + 1$$

$$= p^2 \; -\frac{(-2)}{p^3} \; \cancel{+0}$$

$$= \frac{2}{p}$$

w/ $p = 1/4$ of success

$$= \frac{2}{1/4} = 2(4) = 8$$

Expected to open 8 boxes in order to get 2 toys