EE 131A
Probability
Instructor: Lara Dolecek

Winter 2020 Midterm A
Monday, February 3, 2020

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name:
[redacted]

Your ID Number:
[redacted]

Name of person on your left:
[redacted]

Name of person on your right:
[redacted]

Problem	Score	Possible
1	10	10
2	13	15
3	15	15
4	15	15
5	15	15
6	15	15
7	15	15
Total	98	100

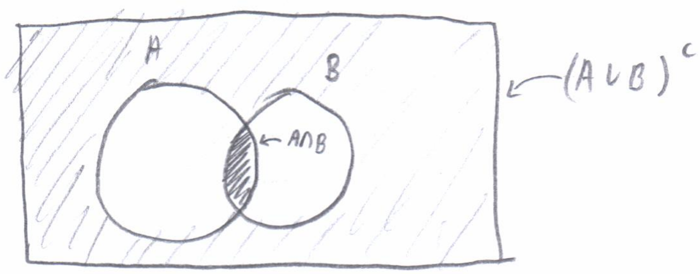
10

1. (5+5 pts)

(a) Draw a Venn Diagram for the events A and B. Shade in the area corresponding to the events $(A \cap B)$ and $(A \cup B)^c$. Clearly indicate which area belongs to which event.

(b) Assume that we throw a single six-sided die. Let A be the event that the result is even and let B be the event that the result is less than or equal to 3. What is $P(A \cap B)$ and $P(A \cup B)$?

a)



b) $S = \{1, 2, 3, 4, 5, 6\}$

$S_A = \{2, 4, 6\}$

$S_B = \{1, 2, 3\}$

$\boxed{P(A \cap B) = \frac{1}{6}}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{6} + \frac{3}{6} - \frac{1}{6}$

$= \boxed{\frac{5}{6}}$

13

2. (15 pts) Let A and B be two events. Given that $P(B) > 0$, prove

$$P(A \cup B^c|B) = P(A \cap B|B).$$

You may use any result taught in lecture or homework.

$$P(A \cup B^c | B) = \frac{P((A \cup B^c) \cap B)}{P(B)}$$

$$= \frac{P((A \cap B) \cup (B^c \cap B))}{P(B)}$$

$$= \frac{P((A \cap B) \cup \emptyset)}{P(B)}$$

?

$$= \frac{P(A \cap B)}{P(B)}$$

−2

$$= P(A \cap B | B) \checkmark$$

15

3. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) For a random variable X, VAR$(aX) = a$VAR(X) for all real values of a.

TRUE FALSE

(b) $P(A|B) = P(B|A)$ holds if and only if the events A and B are mutually exclusive.

TRUE FALSE

(c) A discrete random variable has jump discontinuities in its cumulative distribution function.

TRUE FALSE

(d) If events X and Y are mutually exclusive, then they are also independent.

TRUE FALSE

(e) For random variables X and Y, if E$[XY]$ = E$[X]$E$[Y]$, then X and Y are uncorrelated.

TRUE FALSE

a) $VAR(aX) = E[(aX)^2] - (E[aX])^2$

$= a^2 E[X^2] - a^2 E[X]^2$

$= a^2 (E[X^2] - E[X]^2)$

$= a^2 \, VAR(X)$

b) $\dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A \cap B)}{P(A)}$

$\Rightarrow P(B) = P(A)$

15

4. (3+5+7 pts) Assume there are 5 jars numbered 1 to 5. The ith jar contains i black balls, $6-i$ red balls, and 5 green balls. A jar is selected uniformly at random and a ball is selected from that jar. Let the events B, R, and G represent the events that a black, red, or green ball is chosen, respectively. Let A_k represent the event that the kth jar is chosen.

   (a) What is $P(B|A_k)$? Write the answer in terms of k.

   (b) What is $P(G)$, $P(B)$, and $P(R)$?

   (c) Given that the ball selected was black, what is the probability that the ball came from the kth jar, i.e. $P(A_k|B)$? Write the answer in terms of k.

a) kth jar: k black balls, 6-k red balls, 5 green balls

$\Rightarrow$ 11 total balls

$P(B|A_k) = \boxed{\frac{k}{11}}$ ✓

b) $P(G) = P(G|A_1)\cdot P(A_1) + \dots + P(G|A_5)\cdot P(A_5)$

$= \frac{5}{11}\cdot\frac{1}{5} + \frac{5}{11}\cdot\frac{1}{5} + \dots + \frac{5}{11}\cdot\frac{1}{5}$

$= \frac{1}{11}\cdot 5 \Rightarrow \boxed{P(G) = \frac{5}{11}}$ ✓

$P(B) = P(B|A_1)\cdot P(A_1) + \dots + P(B|A_5)\cdot P(A_5)$

$= \frac{1}{11}\cdot\frac{1}{5} + \frac{2}{11}\cdot\frac{1}{5} + \frac{3}{11}\cdot\frac{1}{5} + \frac{4}{11}\cdot\frac{1}{5} + \frac{5}{11}\cdot\frac{1}{5}$

$= \frac{15}{55} \Rightarrow \boxed{P(B) = \frac{3}{11}}$ ✓

$P(R) = P(R|A_1)\cdot P(A_1) + \dots + P(R|A_5)\cdot P(A_5)$

$= \frac{5}{11}\cdot\frac{1}{5} + \frac{4}{11}\cdot\frac{1}{5} + \frac{2}{11}\cdot\frac{1}{5} + \frac{2}{11}\cdot\frac{1}{5} + \frac{1}{11}\cdot\frac{1}{5}$

$= \frac{15}{55} \Rightarrow \boxed{P(R) = \frac{3}{11}}$ ✓

c) $P(A_k|B)$

$= \frac{P(B|A_k)\cdot P(A_k)}{P(B)}$

$= \frac{\frac{k}{11}\cdot\frac{1}{5}}{\frac{3}{11}}$

$= \frac{\frac{k}{55}}{\frac{3}{11}} = \frac{k}{\cancel{55}_{5}}\cdot\frac{\cancel{11}}{3}$

$= \boxed{\frac{k}{15}}$ ✓

15

5. (15 pts) We select two distinct numbers (a, b) in the range 1 to 99 (inclusive). How many ways can we pick a and b such that their sum is even and a is a multiple of 9?

$1 \sim 99 \Rightarrow$ 50 odd #s and 49 even #s

$S_a = \{9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99\}$

$\Rightarrow$ 6 odd #s, 5 even #s

For sum to be even, we must consider two cases:

(1) a is odd $\Rightarrow$ b must also be odd

Out of 50 total odd numbers, a is odd, so there are 49 remaining odd numbers that b can be. (because the #s must be distinct)

✓

$\Rightarrow$ There are 49×6 ways to pick a and b if a is odd

(2) a is even $\Rightarrow$ b must also be even.

Similarly, there are 48 remaining even numbers for b

✓

$\Rightarrow$ There are 48×5 ways to pick a and b if a is even

In total, there are $294 + 240 = \boxed{534 \text{ ways}}$ ✓

15

6. (~~7+8~~ pts) Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with parameters p_1 for even i and p_2 for odd i where $p_2 \neq p_1$ and n is a positive even number. Let Y be the sum of all the X_i's.

(a) Is Y a Binomial Random Variable? Justify your answer.

(b) Compute E[Y].

a) No. While Binomial RVs can be written as sums of Bernoulli RVs, they take in two parameters: (n, p). This means that all of $X_1, X_2, \ldots, X_n$ must have the same probability for Y to be Binomial, but we know that $p_2 \neq p_1$, so this is not the case.

b) $\mathbb{E}[Y] = \sum_{i=1}^{n} \mathbb{E}[X_i]$

For even i, $p = p_1$, so $\mathbb{E}[X_i] = p_1$

For odd i, $p = p_2$, so $\mathbb{E}[X_i] = p_2$

Because n is even, there are $\frac{n}{2}$ Bernoulli RVs w/ parameter p_1 and $\frac{n}{2}$ with parameter p_2

$$\Rightarrow \mathbb{E}[Y] = \frac{n}{2} \cdot p_1 + \frac{n}{2} \cdot p_2$$

$$= \boxed{n \cdot \frac{p_1 + p_2}{2}}$$

15

7. (3+7+5 pts) Suppose that the continuous random variable X has pdf

$$f(x) = \begin{cases} cx^2 & |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the pdf is valid.

(b) Find E[X] and VAR(X).

(c) Find $P(X \ge \frac{1}{2})$.

a) $\int_{-\infty}^{\infty} f_X(t)\,dt = 1$

$\Rightarrow \int_{-1}^{1} cx^2\,dx = 1$

$\frac{c}{3}[x^3]_{-1}^{1} = 1$

$\frac{c}{3}[1-(-1)] = 1$

$\Rightarrow \boxed{c = \frac{3}{2}}$

b) $E[X] = \int_{-\infty}^{\infty} x f_x(t)\,dt$

$= \frac{3}{2}\int_{-1}^{1} x^3\,dx$

$= \frac{3}{2}\cdot\frac{1}{4}[x^4]_{-1}^{1}$

$= \frac{3}{8}[1-1]$

$\boxed{E[X] = 0}$

$VAR(X) = E[X^2] - (E[X])^2$

$= \frac{3}{2}\int_{-1}^{1} x^4\,dx - 0^2$

$= \frac{3}{2}\cdot\frac{1}{5}(x^5)\Big|_{-1}^{1}$

$= \frac{3}{10}\cdot 2 \Rightarrow \boxed{VAR(X) = \frac{3}{5}}$

c) $P(X \ge \frac{1}{2}) = P(\frac{1}{2} \le X < \infty) = P(\frac{1}{2} \le X \le 1)$

$= \int_{\frac{1}{2}}^{1} \frac{3}{2}x^2\,dx$

$= \frac{3}{2}\cdot\frac{1}{3}[x^3]_{\frac{1}{2}}^{1}$

$= \frac{1}{2}(1-\frac{1}{8})$

$= \frac{1}{2}\cdot\frac{7}{8} = \boxed{\frac{7}{16}}$